14.3 Partial Derivatives

14.3.1 Partial Derivatives

Definition 1. *The partial derivative of f with respect to x is*

$$f_x(x, y) = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h}$$

The partial derivative of f with respect to y is

$$f_y(x, y) = \lim_{h \to 0} \frac{f(x, y+h) - f(x, y)}{h}$$

Remark. *Notations for partial derivatives are:*

$$f_x(x, y) = f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} f(x, y) = \frac{\partial z}{\partial x} = f_1 = D_1 f = D_x f$$
$$f_y(x, y) = f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y} f(x, y) = \frac{\partial z}{\partial y} = f_2 = D_2 f = D_y f$$

1. To find f_x, we regard y as a constant and differentiate $f(x, y)$ with respect to x.
2. To find f_y, we regard x as a constant and differentiate $f(x, y)$ with respect to y.

Example 1. *Find the first partial derivatives of the function*

$$u(r, \theta) = \sin(r \cos \theta).$$

14.3.2 Interpretations of Partial Derivatives

We may interpret the partial derivatives as slopes or as rates of change. If $z = f(x, y)$, then $\partial z/\partial x$ represents the rate of change of z with respect to x when y is fixed. Similarly, $\partial z/\partial y$ represents the rate of change of z with respect to y when x is fixed.

Example 2. *Assuming $z = f(x, y)$, use implicit differentiation to find $\partial z/\partial x$ and $\partial z/\partial y$.*

$$yz + x \ln y = z^2$$

14.3.3 Functions of More Than Two Variables

We may define partial derivatives for functions of more than two variables similarly to functions of two variables. For example,

$$f_x(x, y, z) = \lim_{h \to 0} \frac{f(x+h, y, z) - f(x, y, z)}{h}$$

14.3.4 Higher Derivatives

If $z = f(x, y)$, we have

$$(f_x)_x = f_{xx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 z}{\partial x^2}$$
$$(f_x)_y = f_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 z}{\partial y \partial x}$$

Note that in $f_{xy} = \frac{\partial^2 f}{\partial y \partial x}$ we first differentiate with respect to x and then with respect to y. In f_{yx} the order of differentiation is reversed.

Example 3. *If $u = x^a y^b z^c$, then find*

$$\frac{\partial^6 u}{\partial x \partial y^2 \partial z^3}$$

Theorem (Clairaut's Theorem)**.** *Suppose f is defined on a disk D that contains the point (a, b). If the functions f_{xy} and f_{yx} are both continuous on D, then*

$$f_{xy}(a, b) = f_{yx}(a, b).$$

Thus we have $f_{xy} = f_{yx}$ for most functions that we encounter.

14.3.5 Partial Differential Equations

Partial differential equations are equations that contain partial derivatives. The following two partial differential equations are useful in applications.

1. $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

   is called Laplace's equation. Solutions of Laplace's equation are called harmonic functions. Harmonic functions are useful in solving problems of heat conduction, fluid flow, and electric potential.

2. $$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}$$

   is called the wave equation. The wave equation describes the motion of a waveform, such as an ocean wave, a sound wave, or a light wave. If $u(x, t)$ is the displacement of a vibrating violin string at time t at a distance x from one end of the string, then $u(x, t)$ is a solution to the wave equation. Here the constant a depends on the density of the string and on the tension in the string.

Example 4. *Determine whether each of the following functions is a solution of Laplace's equation $u_{xx} + u_{yy} = 0$.*

a) $u = x^2 + y^2$

b) $u = e^{-x} \cos y - e^{-y} \cos x$

Example 5. *Show that each of the following functions is a solution of the wave equation $u_{tt} = a^2 u_{xx}$.*

a) $u = \sin(kx) \sin(akt)$

b) $u = \sin(x - at) + \ln(x + at)$