14.2 Limits and Continuity

14.2.1 Limits

We use the notation

$$\lim_{(x,y)\to(a,b)} f(x,y) = L$$

to indicate that the values of $f(x,y)$ approach the number L as the point (x,y) approaches the point (a,b) along any path that stays within the domain of f.

The limit laws for functions of one variable may be extended to functions of two variables. For example, the limit of a sum is the sum of limits, and the limit of a product is the product of limits. In particular, we have

$$\lim_{(x,y)\to(a,b)} x = a, \qquad \lim_{(x,y)\to(a,b)} y = b, \qquad \lim_{(x,y)\to(a,b)} c = c$$

The Squeeze Theorem also holds.

For a function of a single variable, there are only two possible directions of approach: from the left or from the right. For functions of two variables, there may be infinite number of directions as long as (x,y) stays within the domain of f. By definition, the distance between $f(x,y)$ and L can be made arbitrarily small by making the distance from (x,y) to (a,b) sufficiently small (but not 0). Note that we refer only to the *distance*. So if the limit exists, then $f(x,y)$ must approach the same limit no matter how (x,y) approaches (a,b). Therefore, if we can find two different paths of approach along which the function f has different limits, then the limit does not exist.

Remark. *If $f(x,y) \to L_1$ as $(x,y) \to (a,b)$ along a path C_1 and $f(x,y) \to L_2$ as $(x,y) \to (a,b)$ along a path C_2, where $L_1 \neq L_2$, then $\lim_{(x,y)\to(a,b)} f(x,y)$ does not exist.*

14.2.2 Continuity

A function f of two variables is called *continuous* at (a,b) if

$$\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$$

Thus evaluating limits of continuous functions is easy: just directly substitute the values into the function definition. Intuitively, the surface that is the graph of a continuous function has no hole or break. Using the properties of limits, the differences, products, and quotients of continuous functions are also continuous on their domains. Thus a polynomial function of two variables, which is a sum of terms of the form $cx^m y^n$ for a constant c and nonnegative integers m and n, is also continuous. So is a rational function, which is the ratio of two polynomials.

Example 1. *Find the limit, if it exists, or show that the limit does not exist.*

a) $\lim_{(x,y)\to(2,-1)} \frac{x^2y+xy^2}{x^2-y^2}$

b) $\lim_{(x,y)\to(3,2)} e^{\sqrt{2x-y}}$

c) $\lim_{(x,y)\to(0,0)} \frac{5y^4\cos^2 x}{x^4+y^4}$

d) $\lim_{(x,y)\to(0,0)} \frac{x^3-y^3}{x^2+xy+y^2}$

e) $\lim_{(x,y)\to(0,0)} \frac{xy^4}{x^4+y^4}$

f) $\lim_{(x,y)\to(0,0)} \frac{xy^4}{x^2+y^8}$

If (r,θ) are polar coordinates of the point (x,y) with $r \geq 0$, then $r \to 0^+$ as $(x,y) \to (0,0)$.

Example 2. *Use polar coordinates to find the limit.*

$$\lim_{(x,y)\to(0,0)} (x^2+y^2)\ln(x^2+y^2)$$

14.2.3 Functions of Three or More Variables

We may extend everything that we have done in this section to functions of three or more variables. So for example, f is continuous at (a, b, c) if

$$\lim_{(x,y,z)\to(a,b,c)} f(x, y, z) = f(a, b, c)$$

We may use the vector notation to define the limit of a function of n variables as the limit of a function of a *vector* $\vec{x} = \langle x_1, x_2, \ldots, x_n\rangle$. Thus $\lim_{\vec{x}\to\vec{a}} f(\vec{x}) = L$ means that for every number $\epsilon > 0$, there is a corresponding number $\delta > 0$ such that

$$0 < |\vec{x} - \vec{a}| < \delta \Rightarrow |f(\vec{x}) - L| < \epsilon$$

And f is continuous at $\vec{a}$ if and only if

$$\lim_{\vec{x}\to\vec{a}} f(\vec{x}) = f(\vec{a})$$