13 Vector Functions

13.1 Vector Functions and Space Curves

A *vector function* is a function with an input of a real number and an output that is a vector. For example, $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$ is a vector function, where t is the real-number input and f, g, h are functions of t. Usually the input is the time, hence the choice of the letter t.

13.1.1 Limits and Continuity

If $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$, then

$$\lim_{t \to a} \vec{r}(t) = \langle \lim_{t \to a} f(t), \lim_{t \to a} g(t), \lim_{t \to a} h(t) \rangle$$

provided the limits of the component functions exist.

13.1.2 Space Curves

Suppose that f, g, h are continuous functions. Then the set of all points $C = \{(x, y, z) : (x, y, z) = (f(t), g(t), h(t))\}$ is called a *space curve*. The parametric equations of the space curve C are

$$x = f(t), \quad y = g(t), \quad z = h(t)$$

where the parameter is t.

Example 1. *Sketch the curve with the vector equation*

$$\vec{r}(t) = 2\cos t\hat{\imath} + 2\sin t\hat{\jmath} + \hat{k}$$

Indicate with an arrow the direction in which t increases.

Since space curves are usually difficult to draw by hand, we may use a software program to generate the curve.

13.2 Derivatives and Integrals of Vector Functions

13.2.1 Derivatives

We define

$$\frac{d\vec{r}}{dt} = \vec{r'}(t) = \lim_{h \to 0} \frac{\vec{r}(t+h) - \vec{r}(t)}{h}$$

if the limit exists. The vector $\vec{r'}(t)$ is called the *tangent vector* to the curve defined by $\vec{r}$, provided $\vec{r'}(t)$ exists and $\vec{r'}(t) \neq \vec{0}$. The *unit tangent vector* is

$$\vec{T}(t) = \frac{\vec{r'}(t)}{|\vec{r'}(t)|}$$

Theorem. *If $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$, where f, g, h are differentiable functions, then*

$$\vec{r'}(t) = \langle f'(t), g'(t), h'(t) \rangle$$

We may prove the theorem by the limit definition of the derivative.

13.2.2 Differentiation Rules

Theorem. *Suppose $\boldsymbol{u}$ and $\boldsymbol{v}$ are differentiable vector functions, c is a scalar, and f is a real-valued function. Then*

1. $\frac{d}{dt}[\boldsymbol{u}(t) + \boldsymbol{v}(t)] = \boldsymbol{u}'(t) + \boldsymbol{v}'(t)$
2. $\frac{d}{dt}[c\boldsymbol{u}(t)] = c\boldsymbol{u}'(t)$
3. $\frac{d}{dt}[f(t)\boldsymbol{u}(t)] = f'(t)\boldsymbol{u}(t) + f(t)\boldsymbol{u}'(t)$
4. $\frac{d}{dt}[\boldsymbol{u}(t) \cdot \boldsymbol{v}(t)] = \boldsymbol{u}'(t) \cdot \boldsymbol{v}(t) + \boldsymbol{u}(t) \cdot \boldsymbol{v}'(t)$
5. $\frac{d}{dt}[\boldsymbol{u}(t) \times \boldsymbol{v}(t)] = \boldsymbol{u}'(t) \times \boldsymbol{v}(t) + \boldsymbol{u}(t) \times \boldsymbol{v}'(t)$
6. $\frac{d}{dt}[\boldsymbol{u}(f(t))] = f'(t)\boldsymbol{u}'(f(t))$ *(Chain Rule)*

Example 2. *Find parametric equations for the tangent line to the curve with the parametric equations*

$$x = \ln(t+1), \quad y = t\cos 2t, \quad z = 2^t$$

at the point $(0, 0, 1)$.

13.2.3 Integrals

We define

$$\int_a^b \vec{r}(t)dt = \left\langle \left(\int_a^b f(t)dt \right), \left(\int_a^b g(t)dt \right), \left(\int_a^b h(t)dt \right) \right\rangle$$