12.5 Equations of Lines and Planes

12.5.1 Lines

In $\mathbb{R}^2$, a line is determined by its slope (direction) and a point, like the y-intercept, or the initial value. In $\mathbb{R}^3$, we may also determine a line by its direction and an initial point. If the line is parallel to a vector $\vec{v}$ and passing through an initial point depicted by a vector $\vec{r}_0$, then each point on the line is pointed at by a vector $\vec{r}$ such that

$$\vec{r} = \vec{r}_0 + t\vec{v}$$

where t is a real number. This is the vector equation of the line.

If $\vec{r} = \langle x, y, z\rangle, \vec{r}_0 = \langle x_0, y_0, z_0\rangle$, and $\vec{v} = \langle a, b, c\rangle$, then

$$\langle x, y, z\rangle = \langle x_0, y_0, z_0\rangle + t\langle a, b, c\rangle$$

Then

$$x = x_0 + ta, \quad y = y_0 + tb, \quad z = z_0 + tc$$

These are the parametric equations of the line, with the parameter t. We may solve for t in each parametric equation to obtain

$$t = \frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

These are the symmetric equations of the line. The numbers a, b, c are called the direction numbers of the line. If any of the direction numbers is zero, we may still write the symmetric equations from the parametric equations. For example, if $b = 0$, then we would have $y = y_0$ as the second equation.

The line segment from $\vec{r}_0$ to $\vec{r}_1$ may be written as

$$\vec{r}(t) = (1 - t)\vec{r}_0 + t\vec{r}_1, 0 \le t \le 1$$

Skew lines do not intersect and are not parallel.

Example 1. *Find a vector equation and parametric equations for the line through the point* $(0, 14, -10)$ *and parallel to the line*

$$x = -1 + 2t, y = 6 - 3t, z = 3 + 9t$$

12.5.2 Planes

For a plane, it is not enough to know a vector parallel to the plane to uniquely determine the plane. However, a vector perpendicular to the plane, together with a point on the plane, uniquely define the plane. This perpendicular vector is called the *normal vector*.

If we have a point $P_0(x_0, y_0, z_0)$ on the plane and $P(x, y, z)$ is any point on the plane, let $\vec{r}_0$ be the position vector of P_0 and $\vec{r}$ be the position vector for P. Then the normal vector $\vec{n}$ to the plane is orthogonal to $\vec{r} - \vec{r}_0$. That is,

$$\vec{n} \cdot (\vec{r} - \vec{r}_0) = 0$$

Or

$$\vec{n} \cdot \vec{r} = \vec{n} \cdot \vec{r}_0$$

These are the vector equations of the plane.

If $\vec{n} = \langle a, b, c\rangle, \vec{r} = \langle x, y, z\rangle$, and $\vec{r}_0 = \langle x_0, y_0, z_0\rangle$, then the scalar equation of the plane through $P_0(x_0, y_0, z_0)$ with normal vector $\vec{n}$ is

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

We may write the above equation as

$$ax + by + cz + d = 0$$

where $d = -(ax_0 + by_0 + cz_0)$. This is the *linear equation* in x, y, z of the plane.

Two planes are either parallel or they intersect. Two planes are parallel if and only if their normal vectors are parallel, that is, their normal vectors are nonzero multiples of each other. If two planes are not parallel, they intersect at a line. We define the angle between the two planes as the acute angle between their normal vectors. We may regard the line

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

as the line of intersection of the two planes

$$\frac{x - x_0}{a} = \frac{y - y_0}{b}$$

and

$$\frac{y - y_0}{b} = \frac{z - z_0}{c}$$

Example 2. *Find the equation of the plane through the origin and the points* $(3, -2, 1)$ *and* $(1, 1, 1)$.

12.5.3 Distances

If $P_0(x_0, y_0, z_0)$ is any point on a plane, the distance D from a point $P_1(x_1, y_1, z_1)$ to the plane is the scalar projection of the vector $\overrightarrow{P_0P_1}$ onto the normal vector $\vec{n} = \langle a, b, c \rangle$. If the angle between $\overrightarrow{P_0P_1}$ and $\vec{n}$ is θ, then the distance D is

$$\begin{aligned} D &= |\overrightarrow{P_0P_1}||\cos\theta| \\ &= \frac{|\vec{n}||\overrightarrow{P_0P_1}||\cos\theta|}{|\vec{n}|} \\ &= \frac{|\vec{n} \cdot \overrightarrow{P_0P_1}|}{|\vec{n}|} \\ &= \frac{|a(x_1 - x_0) + b(y_1 - y_0) + c(z_1 - z_0)|}{\sqrt{a^2 + b^2 + c^2}} \\ &= \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}} \end{aligned}$$

where $d = -(ax_0 + by_0 + cz_0)$.

Example 3. *Find the distance between the skew lines with parametric equations*

$$x = 1 + t, y = 1 + 6t, z = 2t$$

and

$$x = 1 + 2s, y = 5 + 15s, z = -2 + 6s$$