12.4 The Cross Product

12.4.1 2×2 Determinant

We define the 2×2 determinant

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1 b_2 - a_2 b_1$$

That is, the product of the entries on the main diagonal minus the product of the entries on the other diagonal.

12.4.2 Cross Product

Suppose we want to find a vector $\vec{n} = \langle a, b, c \rangle$ perpendicular to two given nonzero vectors $\vec{x}_1 = \langle x_1, y_1, z_1 \rangle$ and $\vec{x}_2 = \langle x_2, y_2, z_2 \rangle$. Since

$$\langle x_1, y_1, z_1 \rangle \cdot \langle a, b, c \rangle = 0 \quad \text{and} \quad \langle x_2, y_2, z_2 \rangle \cdot \langle a, b, c \rangle = 0$$

We seek a, b, and c satisfying the system of equations

$$\begin{aligned} x_1 a + y_1 b + z_1 c &= 0 \\ x_2 a + y_2 b + z_2 c &= 0 \end{aligned}$$

If we solve for a and b in terms of c, by the process of elimination, we multiply the first equation by y_2 and the second equation by y_1, we have

$$\begin{aligned} x_1 y_2 a + y_1 y_2 b &= -y_2 z_1 c \\ x_2 y_1 a + y_1 y_2 b &= -y_1 z_2 c \end{aligned}$$

Subtract the second equation from the first to eliminate b:

$$(x_1 y_2 - x_2 y_1) a = (y_1 z_2 - y_2 z_1) c$$

Assuming $x_1 y_2 - x_2 y_1 \neq 0$, we find

$$a = \left(\frac{y_1 z_2 - y_2 z_1}{x_1 y_2 - x_2 y_1} \right) c$$

Eliminating a from the original system in the same way, we find

$$b = \left(\frac{x_2 z_1 - x_1 z_2}{x_1 y_2 - x_2 y_1} \right) c$$

Notice that the numerators and the denominators are 2×2 determinants. Now suppose that we choose the convenient value

$$c = \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} = x_1 y_2 - x_2 y_1.$$

Then the formulas for a and b reduce to

$$a = \begin{vmatrix} y_1 & z_1 \\ y_2 & z_2 \end{vmatrix}, \quad b = - \begin{vmatrix} x_1 & z_1 \\ x_2 & z_2 \end{vmatrix}$$

This vector perpendicular to both $\vec{x}_1$ and $\vec{x}_2$ is called the *cross product* of $\vec{x}_1$ and $\vec{x}_2$, written

$$\vec{n} = \vec{x}_1 \times \vec{x}_2$$

A convenient way to remember the formula for $\vec{n}$ is to write

$$\vec{x}_1 \times \vec{x}_2 = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix}$$

12.4.3 3×3 Determinant

The third-order determinant

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

is defined to be the number

$$a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

This expression may be described as the sum of the entries of the first row multiplied by their *cofactors*

$$\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}, - \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix}, \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

Notice the patter $+, -, +$. There are other ways to "expand" a third-oder determinant. Our purpose, however, is not to discuss determinants, but to explain the formula

$$\vec{x}_1 \times \vec{x}_2 = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix}$$

This is a "pseudo-determinant," in the sense that the first row is not a triple of numbers, but the vector triple $\hat{\imath}, \hat{\jmath}, \hat{k}$. The pattern of expansion is the same, however.

Theorem. *Assuming $\vec{x}_1$ and $\vec{x}_2$ are not zero, the magnitude of the cross product is*

$$|\vec{x}_1 \times \vec{x}_2| = |\vec{x}_1||\vec{x}_2||\sin\theta|, 0 \le \theta \le \pi,$$

where θ is the angle between $\vec{x}_1$ and $\vec{x}_2$.

Corollary.

$$\vec{x}_1 \parallel \vec{x}_2 \Leftrightarrow \vec{x}_1 \times \vec{x}_2 = \vec{0}.$$

for nonzero vectors $\vec{x}_1$ and $\vec{x}_2$.

Note that we may also use the test for parallel vectors as $|\vec{x}_1 \cdot \vec{x}_2| = |\vec{x}_1||\vec{x}_2|$.

Remark. *The cross product is "anticommutative," a term that means*

$$\vec{x}_1 \times \vec{x}_2 = -\vec{x}_2 \times \vec{x}_1$$

Associative law also fails.

$$(\vec{x}_1 \times \vec{x}_2) \times \vec{x}_3 \ne \vec{x}_1 \times (\vec{x}_2 \times \vec{x}_3)$$

However, the distributive law holds.

$$\vec{x}_1 \times (\vec{x}_2 + \vec{x}_3) = (\vec{x}_1 \times \vec{x}_2) + (\vec{x}_1 \times \vec{x}_3)$$

Remark. *If the cross product of two vectors $\vec{x}_1$ and $\vec{x}_2$ is not zero, then $\vec{x}_1$ and $\vec{x}_2$ determine a plane. The vector $\vec{n} = \vec{x}_1 \times \vec{x}_2$ is orthogonal to the plane and is directed so that the triple $\vec{x}_1, \vec{x}_2, \vec{n}$ is right-handed.*

Remark. *Geometrically, if $\vec{x}_1$ and $\vec{x}_2$ determine a parallelogram with base $|\vec{x}_1|$ and altitude $|\vec{x}_2 \sin\theta|$, then the length of the cross product $\vec{x}_1 \times \vec{x}_2$ is equal to the area of the parallelogram determined by $\vec{x}_1$ and $\vec{x}_2$.*

Example 1.

$$\hat{\imath} \times \hat{\jmath} = \hat{k}$$
$$\hat{\jmath} \times \hat{k} = \hat{\imath}$$
$$\hat{k} \times \hat{\imath} = \hat{\jmath}$$
$$\hat{\jmath} \times \hat{\imath} = -\hat{k}$$

12.4.4 Triple Product

There is a special scalar triple product property:

$$(\vec{a} \times \vec{b}) \cdot \vec{c} = \vec{a} \cdot (\vec{b} \times \vec{c}).$$

We may write the scalar triple product as a determinant:

$$\vec{a} \cdot (\vec{b} \times \vec{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Geometrically, two vectors $\vec{b}$ and $\vec{c}$ determine a parallelogram with area $|\vec{b}||\vec{c}||\sin\theta|$. All three vectors form a parallelepiped with height $|\vec{a}||\cos\theta|$. Thus we have a formula for the volume of the parallelepiped:

$$V = Ah = |\vec{b} \times \vec{c}||\vec{a}||\cos\theta| = |\vec{a} \cdot (\vec{b} \times \vec{c})|.$$

Example 2. *If* $\boldsymbol{a} = \langle 1, 0, 1\rangle, \boldsymbol{b} = \langle 2, 1, -1\rangle, \boldsymbol{c} = \langle 0, 1, 3\rangle$*, show that* $\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) \neq (\boldsymbol{a} \times \boldsymbol{b}) \times \boldsymbol{c}$.

Example 3. *Consider the three points* $P(0, 0, -3), Q(4, 2, 0), R(3, 3, 1)$.

a) Find a nonzero vector orthogonal to the plane through the points P, Q, R.

b) Find the area of the triangle PQR.

Example 4. *Consider the points* $P(-2, 1, 0), Q(2, 3, 2), R(1, 4, -1), S(3, 6, 1)$. *Find the volume of the parallelepiped with adjacent edges* PQ, PR, PS.