12.3 The Dot Product

There is a special way to "multiply" two vectors called the *dot product*. We define the dot product of $\vec{v} = \langle v_1, v_2, v_3 \rangle$ with $\vec{w} = \langle w_1, w_2, w_3 \rangle$ as

$$\vec{v} \cdot \vec{w} = \langle v_1, v_2, v_3 \rangle \cdot \langle w_1, w_2, w_3 \rangle = v_1 w_1 + v_2 w_2 + v_3 w_3$$

Note that the dot product of two vectors is a number, not a vector. Obviously $\vec{v} \cdot \vec{v} = |\vec{v}|^2$ for all vectors $\vec{v} \in \mathbb{R}^n$. In particular, $\vec{v} \cdot \vec{v} \geq 0$ for all vectors $\vec{v}$, with equality if and only if $\vec{v} = 0$. Furthermore, $\vec{v} \cdot \vec{w} = \vec{w} \cdot \vec{v}$. Other properties of the dot product are that for all vectors $\vec{v}, \vec{w}, \vec{u}$ and for all scalars a, we have

$$\begin{aligned} (\vec{u} + \vec{v}) \cdot \vec{w} &= \vec{u} \cdot \vec{w} + \vec{v} \cdot \vec{w} \\ (a\vec{v}) \cdot \vec{w} &= a(\vec{v} \cdot \vec{w}) \end{aligned}$$

Two vectors $\vec{u}, \vec{v}$ are said to be ***orthogonal*** if $\vec{u} \cdot \vec{v} = 0$.

$$\vec{v} \perp \vec{w} \Leftrightarrow \vec{v} \cdot \vec{w} = 0.$$

Clearly $\vec{0}$ is orthogonal to every vector. Furthermore, $\vec{0}$ is the only vector that is orthogonal to itself.

The next theorem is over 2,500 years old.

Theorem (Pythagorean Theorem)**.** *If $\vec{u}, \vec{v}$ are orthogonal, then*

$$|\vec{u} + \vec{v}|^2 = |\vec{u}|^2 + |\vec{v}|^2.$$

Proof. Suppose that $\vec{u}, \vec{v}$ are orthogonal vectors. Then

$$\begin{aligned} |\vec{u} + \vec{v}|^2 &= (\vec{u} + \vec{v}) \cdot (\vec{u} + \vec{v}) \\ &= |\vec{u}|^2 + |\vec{v}|^2 + \vec{u} \cdot \vec{v} + \vec{v} \cdot \vec{u} \\ &= |\vec{u}|^2 + |\vec{v}|^2, \end{aligned}$$

as desired. □

Theorem. *If θ is the angle between the vectors $\vec{a}$ and $\vec{b}$, then*

$$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos\theta.$$

Proof. By the equivalence of the law of cosines and the dot product definition of $|\vec{a} - \vec{b}|^2$. □

Corollary. *If θ is the angle between the nonzero vectors $\vec{a}$ and $\vec{b}$, then*

$$\cos\theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|}.$$

Example 1. *Determine whether the following vectors are orthogonal, parallel, or neither:*

a) $\boldsymbol{u} = \langle -5, 4, -2 \rangle, \quad \boldsymbol{v} = \langle 3, 4, -1 \rangle$

b) $\boldsymbol{u} = 9\boldsymbol{i} - 6\boldsymbol{j} + 3\boldsymbol{k}, \quad \boldsymbol{v} = -6\boldsymbol{i} + 4\boldsymbol{j} - 2\boldsymbol{k}$

c) $\boldsymbol{u} = \langle c, c, c \rangle, \quad \boldsymbol{v} = \langle c, 0, -c \rangle$

Example 2. *Find the values of x such that the angle between the vectors $\langle 2, 1, -1 \rangle$ and $\langle 1, x, 0 \rangle$ is 45°.*

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12.3.1 Direction Angles and Direction Cosines

The *direction angles* of a nonzero vector $\vec{v}$ are the angles α, β, and γ, each between 0 and π, that the vector $\vec{v}$ makes with the positive x-, y-, and z-axes, respectively. The cosines of these direction angles, $\cos\alpha, \cos\beta$, and $\cos\gamma$, are called the direction cosines of the vector $\vec{v}$. Using the previous corollary, we have

$$\cos\alpha = \frac{\vec{v}\cdot\hat{\imath}}{|\vec{v}||\hat{\imath}|} = \frac{v_1}{|\vec{v}|}$$

Similarly,

$$\cos\beta = \frac{v_2}{|\vec{v}|} \quad \text{and} \quad \cos\gamma = \frac{v_3}{|\vec{v}|}$$

Therefore

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

Furthermore,

$$\vec{v} = \langle v_1, v_2, v_3\rangle = \langle |\vec{v}|\cos\alpha, |\vec{v}|\cos\beta, |\vec{v}|\cos\gamma\rangle = |\vec{v}|\langle\cos\alpha, \cos\beta, \cos\gamma\rangle$$

Therefore

$$\frac{\vec{v}}{|\vec{v}|} = \langle\cos\alpha, \cos\beta, \cos\gamma\rangle$$

That is, the direction cosines of $\vec{v}$ are the components of the unit vector in the direction of $\vec{v}$.

12.3.2 Projections

The *vector projection* of $\vec{w}$ onto $\vec{v}$, denoted by $\text{proj}_{\vec{v}}\vec{w}$, is the shadow of $\vec{w}$ on $\vec{v}$. The scalar projection of $\vec{w}$ onto $\vec{v}$, also called the component of $\vec{w}$ along $\vec{v}$, is defined to be the signed magnitude of the vector projection, which is the number $|\vec{w}|\cos\theta$, where θ is the angle between $\vec{v}$ and $\vec{w}$. This is denoted by $\text{comp}_{\vec{v}}\vec{w}$. When $\pi/2 < \theta \leq \pi$, we have $\text{comp}_{\vec{v}}\vec{w} < 0$. The equation

$$\vec{v}\cdot\vec{w} = |\vec{v}||\vec{w}|\cos\theta = |\vec{v}|(|\vec{w}|\cos\theta)$$

shows that the dot product of $\vec{v}$ and $\vec{w}$ can be interpreted as the length of $\vec{v}$ times the scalar projection of $\vec{w}$ onto $\vec{v}$. Since

$$|\vec{w}|\cos\theta = \frac{\vec{v}\cdot\vec{w}}{|\vec{v}|} = \frac{\vec{v}}{|\vec{v}|}\cdot\vec{w}$$

the component of $\vec{w}$ along $\vec{v}$ can be computed by taking the dot product of $\vec{w}$ with the unit vector in the direction of $\vec{v}$. Therefore

$$\text{comp}_{\vec{v}}\vec{w} = \frac{\vec{v}\cdot\vec{w}}{|\vec{v}|}$$
$$\text{proj}_{\vec{v}}\vec{w} = \left(\frac{\vec{v}\cdot\vec{w}}{|\vec{v}|}\right)\frac{\vec{v}}{|\vec{v}|} = \frac{\vec{v}\cdot\vec{w}}{|\vec{v}|^2}\vec{v}$$

Example 3. *Find the angle between a diagonal of a cube and a diagonal of one of its faces.*