12 Vectors and the Geometry of Space

12.1 Three-Dimensional Coordinate Systems

12.1.1 3D Space

The multivariable calculus that we study is mostly two or three variables. Two variables define a two-dimensional plane, such as the Cartesian plane. With three variables, we have three dimensions. Let's consider the Cartesian 3D space, with the axes x, y, z. We call the two-dimensional space $\mathbb{R}^2$ and the three dimensional space $\mathbb{R}^3$. Each point has a unique address in Cartesian space as (x, y, z), which is an ordered triple.

By Pythagorean theorem: the distance $|P_1P_2|$ between points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ is derived as follows:

$$|P_1P_2|^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2$$

So the equation of a sphere with center $C(h, k, l)$ and radius r is

$$(x - h)^2 + (y - k)^2 + (z - l)^2 = r^2$$

In particular, the sphere with center at origin has the equation $x^2 + y^2 + z^2 = r^2$.

Example 1. *Find an equation of the sphere that passes through the origin and whose center is* $(1, 2, 3)$.

Example 2. *Describe in words the region of* $\mathbb{R}^3$ *represented by the equation* $y = -2$.

Example 3. *Describe in words the region of* $\mathbb{R}^3$ *represented by the inequality* $x^2 + y^2 + z^2 \leq 4$.

12.2 Vectors

Let's look at two important examples of a vector space.

The vector space $\mathbb{R}^2$, which you can think of as a plane, consists of all ordered pairs of real numbers:

$$\mathbb{R}^2 = \{(x, y) : x, y \in \mathbb{R}\}.$$

The vector space $\mathbb{R}^3$, which you can think of as ordinary space, consists of all ordered triples of real numbers:

$$\mathbb{R}^3 = \{(x, y, z) : x, y, z \in \mathbb{R}\}.$$

We may generalize $\mathbb{R}^2$ and $\mathbb{R}^3$ to higher dimensions as $\mathbb{R}^n$. If $n \geq 4$, we cannot easily visualize $\mathbb{R}^n$ as a physical object. That would be the topic of a different course, namely linear algebra. So we only consider $\mathbb{R}^2$ and $\mathbb{R}^3$.

A typical element of $\mathbb{R}^2$ is a point $v = (x, y)$. Sometimes we think of v not as a point, but as an arrow starting at the origin and ending at (x, y). When we think of v as an arrow, we refer to it as a ***vector*** and write $\vec{v} = \langle x, y \rangle$. x and y are called the *components* of $\vec{v}$.

Often you will gain better understanding by dispensing with the coordinate axes and just thinking of the vector as an arrow.

Addition has a simple geometric interpretation. Suppose we have two vectors $\vec{v}$ and $\vec{w}$ in $\mathbb{R}^2$ that we want to add. Move the vector $\vec{w}$ parallel to itself so that its initial point coincides with the end point of the vector $\vec{v}$. The sum $\vec{v} + \vec{w}$ then equals the vector whose initial point equals the initial point of $\vec{v}$ and whose end point equals the end point of the moved vector $\vec{w}$.

Our treatment of the vector $\vec{w}$ above illustrates a standard philosophy when we think of vectors in $\mathbb{R}^2$ as arrows: we can move an arrow parallel to itself (not changing its length or direction) and still think of it as the same vector.

We define what it means to multiply an element of $\mathbb{R}^n$ by an element of $\mathbb{R}$, such as

$$a\langle x, y, z \rangle = \langle ax, ay, az \rangle.$$

This way we are scaling the vector. We say we have multiplied the vector $\vec{v}$ by the scalar a.

We define a vector space to be a set V along with addition and scalar multiplication on V. By ***addition*** on V we mean a function that assigns an element $\vec{u}+\vec{v} \in V$ to each pair of elements $\vec{u}, \vec{v} \in V$. By a ***scalar multiplication*** on V we mean a function that assigns an element $a\vec{v} \in V$ to each $a \in \mathbb{R}$ and each $\vec{v} \in V$.

Now we are ready to give a formal definition of a vector space. A ***vector space*** is a set V along with an addition on V and a scalar multiplication on V such that the following properties hold:

commutativity

$$\vec{u}+\vec{v}=\vec{v}+\vec{u}, \quad \forall \vec{u}, \vec{v} \in V$$

associativity

$$(\vec{u}+\vec{v})+\vec{w}=\vec{u}+(\vec{v}+\vec{w}) \text{ and } (ab)\vec{v}=a(b\vec{v}), \quad \forall \vec{u}, \vec{v}, \vec{w} \in V \text{ and } \forall a, b \in \mathbb{R}$$

additive identity

$$\exists \vec{0} \in V : \vec{v}+\vec{0}=\vec{v}, \quad \forall \vec{v} \in V$$

additive inverse

$$\forall \vec{u}, \vec{v} \in V, \quad \exists \vec{w} \in V : \vec{v}+\vec{w}=\vec{0}$$

multiplicative identity

$$1\vec{v}=\vec{v}, \quad \forall \vec{v} \in V$$

distributive properties

$$a(\vec{u}+\vec{v})=a\vec{u}+a\vec{v} \text{ and } (a+b)\vec{v}=a\vec{v}+b\vec{v}, \quad \forall \vec{u}, \vec{v} \in V \text{ and } \forall a, b \in \mathbb{R}$$

We define addition in $\mathbb{R}^n$ by adding corresponding coordinates:

$$\langle v_1, v_2, v_3 \rangle + \langle w_1, w_2, w_3 \rangle = \langle v_1+w_1, v_2+w_2, v_3+w_3 \rangle$$

A *vector* is a mathematical object that has a *length* (magnitude) and a *direction.* We may depict a vector with an arrow above the letter, such as $\vec{v}$. In books, sometimes they use bold typeface for vectors, such as $\mathbf{v}$.

Proposition 1. *A vector space has a unique additive identity.*

Proof. Suppose $\vec{0}$ and $\vec{0}'$ are both additive identities. Then

$$\vec{0}'=\vec{0}'+\vec{0}=\vec{0},$$

where the first equality holds because $\vec{0}$ is an additive identity and the second equality holds because $\vec{0}'$ is an additive identity. Thus $\vec{0}'=\vec{0}$, proving that the additive identity is unique. ☐

Proposition 2. *Every element in a vector space has a unique additive inverse.*

Proof. Suppose $\vec{w}$ and $\vec{w}'$ are additive inverses of $\vec{v}$. Then

$$\vec{w}=\vec{w}+\vec{0}=\vec{w}+(\vec{v}+\vec{w}')=(\vec{w}+\vec{v})+\vec{w}'=\vec{0}+\vec{w}'=\vec{w}'$$

☐

Because additive inverses are unique, we can let $-\vec{v}$ denote the additive inverse of a vector $\vec{v}$. We define $\vec{w}-\vec{v}$ to mean $\vec{w}+(-\vec{v})$.

Proposition 3. $0\vec{v}=\vec{0} \quad \forall \vec{v}$.

Proof.

$$0\vec{v}=(0+0)\vec{v}=0\vec{v}+0\vec{v}.$$

Adding the additive inverse of $0\vec{v}$ to both sides of the equation above gives $\vec{0}=0\vec{v}$, as desired. ☐

Proposition 4. $a\vec{0} = \vec{0} \quad \forall a \in \mathbb{R}$.

Proof.
$$a\vec{0} = a(\vec{0} + \vec{0}) = a\vec{0} + a\vec{0}.$$
Adding the additive inverse of $a\vec{0}$ to both sides of the equation above gives $\vec{0} = a\vec{0}$, as desired. $\square$

Proposition 5. $(-1)\vec{v} = -\vec{v} \quad \forall \vec{v} \in V$.

Proof.
$$\vec{v} + (-1)\vec{v} = 1\vec{v} + (-1)\vec{v} = (1 + (-1))\vec{v} = 0\vec{v} = \vec{0}.$$
This equation says that $(-1)\vec{v}$, when added to $\vec{v}$, gives $\vec{0}$. Thus $(-1)\vec{v}$ must be the additive inverse of $\vec{v}$, as desired. $\square$

By Pythagorean theorem,
$$|\langle x, y\rangle| = \sqrt{x^2 + y^2}$$
and
$$|\langle x, y, z\rangle| = \sqrt{x^2 + y^2 + z^2}$$
We call a vector a *unit* vector if the length of the vector is one. The notation for the length of a vector $\vec{v}$ is $|\vec{v}|$, that is, the same notation as an absolute value. So for unit vector $\vec{u}$, we have $|\vec{u}| = 1$. The unit vectors along the the x, y, z axes are represented, respectively, as $\hat{\imath}, \hat{\jmath}, \hat{k}$. These are called *standard basis vectors.*

$$\hat{\imath} = \langle 1, 0, 0\rangle, \quad \hat{\jmath} = \langle 0, 1, 0\rangle, \quad \hat{k} = \langle 0, 0, 1\rangle$$

We have
$$\langle x, y, z\rangle = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$$
and the unit vector in the same direction as $\vec{v}$ is
$$\vec{u} = \frac{1}{|\vec{v}|}\vec{v} = \frac{\vec{v}}{|\vec{v}|}$$
Often working with vectors, when we write them in terms of basis vectors, helps with solving problems.

Example 4. *Find the magnitude of the resultant force and the angle it makes with the positive x-axis between a force of 20 lb with an angle of $45°$ and a force of 16 lb with an angle of $-30°$.*