10.3 Polar Coordinates

René Descartes introduced the Cartesian coordinate system. Newton invented another coordinate system, called *polar*. Sometimes the polar coordinate system is more efficient than the Cartesian system.

Much like the Cartesian that has an origin, the polar has a point called the *pole*. The ray from the pole extending to the right is called the *polar axis* and corresponds to the positive x-axis. We identify each point in the polar plane by the distance from the pole, r, and the angle θ from the polar axis to the line connecting the pole to the point. Thus the polar coordinate of each point is (r, θ). As usual, the positive angles are going counterclockwise from the polar axis. The pole is at $(0, \theta)$ for any angle θ.

In this book, we extend the meaning of a negative number for the first coordinate as being on the opposite side of the pole when $r > 0$. So the points (r, θ) and $(-r, \theta)$ lie π radians apart:

$$(-r, \theta) = (r, \theta + \pi)$$

Note that in polar coordinate system each point may have more than one representation. For example, $(2, \pi/2)$ and $(2, -3\pi/2)$ are the same point.

$$(r, \theta) = (r, 2k\pi + \theta) = (-r, (2k+1)\pi + \theta), \quad k = 0, \pm 1, \pm 2, \dots$$

The conversion between the polar and Cartesian coordinates is via the following formulas:

$$x = r\cos\theta, \quad y = r\sin\theta$$
$$r^2 = x^2 + y^2, \quad \tan\theta = \frac{y}{x}$$

Note that when we convert from Cartesian to polar, we must choose θ so that the point lies in the correct quadrant.

Example 1. *Suppose* $(x, y) = (\sqrt{3}, -1)$.

a) Find polar coordinates (r, θ) *of the point, where* $r > 0$ *and* $0 \le \theta < 2\pi$.

b) Find polar coordinates (r, θ) *of the point, where* $r < 0$ *and* $0 \le \theta < 2\pi$.

10.3.1 Polar Curves

The graph of a polar equation $F(r, \theta) = 0$ consists of all points (r, θ) that satisfy the equation.

Example 2. *A circle with radius* k*, centered at the pole, has the equation* $r = k$*. The equation* $\theta = m$ *is a line passing through the pole.*

Example 3. *Sketch the curve with the polar equation* $r = 1 + 2\cos\theta$ *by first sketching the graph of* r *as a function of* θ *in Cartesian coordinates.*

10.3.2 Symmetry

- If a polar equation does not change when we replace θ by $-\theta$, then the curve is symmetric about the polar axis.
- If a polar equation does not change when we replace r by $-r$, then the curve is symmetric about the pole.
- If a polar equation does not change when we replace θ by $\pi - \theta$, then the curve is symmetric about the line $\theta = \pi/2$.

10.3.3 Tangents to Polar Curves

Thinking of θ as the parameter in $r = f(\theta)$, we have

$$x = r\cos\theta = f(\theta)\cos\theta, \quad y = r\sin\theta = f(\theta)\sin\theta$$

Thus

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy/d\theta}{dx/d\theta} \\ &= \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}\end{aligned}$$

When a tangent is horizontal, $\frac{dy}{dx} = 0$, that is, $\frac{dy}{d\theta} = 0$. Similarly, when a tangent is vertical, $\frac{dx}{d\theta} = 0$.

Example 4. *Show that the curve $r = 2 - \csc\theta$ (a conchoid) has the line $y = -1$ as a horizontal asymptote by showing that $\lim_{r\to\pm\infty} y = -1$. Use this fact to help sketch the conchoid.*

Example 5. *Find the points on the curve $r = 1 - \sin\theta$ where the tangent line is horizontal or vertical.*