10 Parametric Equations and Polar Coordinates

10.1 Curves Defined by Parametric Equations

Consider the following: a bug traveling along a path. The position of the bug at any moment t may be given by the parametric equations

$$x = f(t), y = g(t), a \leq t \leq b$$

t is called the *parameter*. The parameter may be something other than time, such as an angle. If $t \in [a, b]$, then the *initial point* is $(f(a), g(a))$ and the *terminal point* is $(f(b), g(b))$.

Example 1. *For the parametric equation*

$$x = \frac{1}{2}\cos\theta, \quad y = 2\sin\theta, \quad 0 \leq \theta \leq \pi,$$

a) Eliminate the parameter to find a Cartesian equation of the curve.

b) Sketch the curve and indicate with an arrow the direction in which the curve is traced as the parameter increases.

Example 2. *Describe the motion of a particle with position (x, y) as t varies in the interval $\pi/2 \leq t \leq 2\pi$, where $x = 2 + \sin t$ and $y = 1 + 3\cos t$.*

10.2 Calculus with Parametric Equations

10.2.1 Tangent

By chain rule,

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}$$

Thus we have

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}, \frac{dx}{dt} \neq 0$$

and

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{\frac{d}{dt}(dy/dx)}{dx/dt}$$

Example 3. *Find an equation of the tangent to the curve at the point $(x, y) = (2, e)$ by two methods: (a) without eliminating the parameter and (b) by first eliminating the parameter.*

$$x = 1 + \sqrt{t}, \quad y = e^{t^2},$$

10.2.2 Area

$$A = \int_{x=a}^{x=b} y dx = \int_{t=\alpha}^{t=\beta} y(t)x'(t)dt$$

Example 4. *Find the area enclosed by the x-axis and the curve $x = t^3 + 1$, $y = 2t - t^2$.*

10.2.3 Arc Length

$$L = \int_{x=a}^{x=b} ds = \int_{x=a}^{x=b} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx = \int_{t=\alpha}^{t=\beta} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

Example 5. *Find the exact length of the curve $x = e^t - t$, $y = 4e^{t/2}$, $0 \leq t \leq 2$.*

10.2.4 Surface Area

$$S = \int_{x=a}^{x=b} 2\pi y ds = \int_{x=a}^{x=b} 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx = \int_{t=\alpha}^{t=\beta} 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

Example 6. *Find the exact area of the surface obtained by rotating the given curve about the x-axis:*

$$x = 2t^2 + 1/t, \quad y = 8\sqrt{t}, \quad 1 \le t \le 3.$$