5.4 The Law of Sines and the Law of Cosines

If we know three pieces of information about a triangle, we may solve the triangle, that is, find the values of all sides and all angles. The only exception is that knowing the three angles is not enough, because we may have infinitely many similar triangles.

In this section, we use the convention to name the vertices of a triangle with A, B, and C. We use these capital letters to name the angles in the triangle ABC. We name the sides opposite each angle with the same letters, but lowercase. So the side opposite angle A is called a, and so on.

The Law of Sines

The area of the triangle ABC is

$$\frac{1}{2}ab\sin C = \frac{1}{2}ac\sin B = \frac{1}{2}bc\sin A.$$

Thus if we divide all by abc, and multiply by 2, we obtain the law of sines:

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}.$$

Using the Law of Sines

Example 1. *Solve an AAS case.*

Example 2. *Solve an SSA case.*

If the given angle is acute, we may have two solutions. If the given angle is obtuse, we may have only one solution.

The Law of Cosines

We may divide a triangle into two right triangles. Then use Pythagorean Theorem together with trigonometry to derive the law of cosines:

$$c^2 = a^2 + b^2 - 2ab\cos C.$$

In the special case where $C = \frac{\pi}{2}$, the law of cosines becomes the familiar Pythagorean Theorem.

Using the Law of Cosines

Example 3. *Solve a SSS case.*

Example 4. *Solve a SAS case.*

When to Use Which Law

Rather than memorizing the following, you should be able to determine in each situation which law to use. In a triangle:

Use the law of cosines if you know

- the lengths of all three sides;
- the lengths of two sides and the angle between them.

Use the law of sines if you know

- two angles of a triangle and the length of one side;
- the lengths of two sides and an angle other than between these two sides.

Example 5. *Sirius, the brightest star visible from Earth except for our Sun, is* 8.6 *light-years from Earth. Alpha Centauri, the closest star to Earth except for our Sun, is* 4.4 *light-years from Earth. If the angle Sirius-Earth-Alpha Centauri is* 49°*, then how far apart are Sirius and Alpha Centauri?*