3 Exponential Functions, Logarithms, and e

3.1 Logarithms as Inverses of Exponential Functions

Exponential Functions

Definition 1 *Suppose $b > 0$ and x is an irrational number. Then b^x is the number that is approximated by numbers of the form b^r as r takes on rational values that approximate x.*

Recall the key algebraic properties of exponents, for positive numbers a and b and real numbers x and y:

$$b^x b^y = b^{x+y}, \qquad b^{-x} = \frac{1}{b^x},$$

$$(b^x)^y = b^{xy}, \qquad \frac{a^x}{a^y} = a^{x-y},$$

$$a^x b^x = (ab)^x,$$

$$b^0 = 1, \qquad \frac{a^x}{b^x} = \left(\frac{a}{b}\right)^x.$$

Logarithms Base 2

Definition 2 *The* exponential function *with base b is of the form*

$$f(x) = b^x,$$

where $b > 0$ and $b \neq 1$.

With a positive base b, raised to any power, we always get positive numbers, not even zero. Thus the range is the positive numbers.

Remark 1 *The domain of an exponential function is all real numbers. The range is the set of positive numbers.*

Example 1 *Consider the function $y = 2^x$. We may find a few input-output pairs for this function to better grasp how it behaves. Each time x increases by 1, $f(x)$ doubles, because $2^{x+1} = 2 \cdot 2^x$. Note that $y = 2^x$ is* different *from $y = x^2$. While both graphs are curved, the graph of the exponential function $y = 2^x$ is never zero and always increasing for all real values of x, thus* not *a parabola.*

Logarithms with Any Base

When we restrict the domain of a parabola, such as $y = x^2$, we may get a one-to-one function, and hence could get an inverse function. The inverse function for a parabola $y = x^2$ with domain restricted to $[0, \infty)$ is the square root function $x = \sqrt{y}$.

Since the graph of the exponential function $y = b^x$ is increasing or decreasing, it is one-to-one, and thus invertible. The inverse function for an exponential function is called a *logarithm.*

Example 2 *If we have $y = 2^x$, then inverse function is $x = \log_2 y$, read "log base 2 of y" or "logarithm base 2 of y." The input for a logarithm is a positive number and the output is the exponent for the base 2 to get the input.*

Note that we may switch the numbers in the tables for the previous example to obtain the coordinate pairs for the logarithmic function.

Also note that since logarithm is the exponent, we have

$$2^{\log_2 y} = y.$$

Note that since the range of an exponential function is the set of positive numbers, the domain of its inverse function, the logarithm, is only positive numbers. In particular, $\log_2 0$ is undefined, because $2^x \neq 0$ for all real numbers x.

Common Logarithms and the Number of Digits

Definition 3 *Suppose* b *and* y *are positive numbers, with* $b \neq 1$. *Furthermore, suppose* x *is a real number. Then*

$$\log_b y = x \Leftrightarrow b^x = y.$$

Example 3 *If* $b > 0$ *and* $b \neq 1$, *then*

- $\log_b 1 = 0$, *and*
- $\log_b b = 1$.

Remark 2 *Since logarithmic and exponential functions are inverse of each other, we have*

$$b^{\log_b y} = y, \text{ for all positive numbers} y,$$
$$\log_b b^x = x, \text{ for all numbers } x.$$

This is because $f^{-1}(f(x)) = x$ *and* $f(f^{-1}(y)) = y$ *for any function* f *and its inverse function* f^{-1}.

Remark 3 *If the base* $b > 1$, *then the graph of* $y = \log_b x$ *is increasing, because the graph of* $y = b^x$ *is increasing.*

If the base $b < 1$, *then the graph of* $y = \log_b x$ *is decreasing, because the graph of* $y = b^x$ *is decreasing.*

Most applications of logarithms involve bases bigger than 1.

Remark 4 *One of the most frequently used bases is base* 10. *A logarithm with base* 10 *is called a* common logarithm.