2.5 Rational Functions

The Algebra of Rational Functions

Definition 1 *A* rational function r *is of the form*

$$r(x) = \frac{p(x)}{q(x)},$$

where p and q are polynomials, with $q \neq 0$.

The domain of a rational function is the set of all real numbers, except the zeros of q.

Example 1 *(A rational function with a denominator that has zeros.)*

Remark 1 *All the rules about fractions apply to the rational functions.*

Division of Polynomials

We tend to better grasp the value of a rational number when we write it as an integer plus a fraction with a smaller numerator than the denominator.

Example 2 *(A rational number with numerator larger than the denominator.)*

Similarly, sometimes it is more useful to write a rational function as a polynomial plus a rational function such that the numerator is lower degree than the denominator.

Example 3 *(A rational function with numerator larger degree than the denominator.)*

We may divide polynomials using long division, just as with regular integers.

Example 4 *(A rational function with numerator larger degree than the denominator.)*

Remark 2 *We may write the procedure for dividing polynomials as follows.*

1. *Express the highest degree term in the numerator as a single term times the denominator, plus whatever adjustment terms are necessary.*
2. *Simplify the quotient using the numerator as rewritten in step (1).*
3. *Repeat steps (1) and (2) on the remaining rational function until the degree of the numerator is less than the degree of the denominator or the numerator is* 0.

Theorem 1 (Division Algorithm) *If p and q are polynomials with $q \neq 0$, then there exist polynomials G and R such that*

$$\frac{p}{q} = G + \frac{R}{q}$$

and $\deg R < \deg q$ *or $R = 0$. In other words,*

$$p = qG + R.$$

Example 5 *Suppose $q(x) = x - r$. Then* $\deg R = 0$*, so $R = c$, where c is a constant. Thus*

$$p(x) = (x - r)G(x) + c,$$

or

$$p(x) = (x - r)G(x) + p(r).$$

Thus we have proven the following proposition.

Theorem 2 (Zeros and Factors of a Polynomial) *Suppose p is a polynomial and r is a real number. Then r is a root of p if and only if $x - r$ is a factor of $p(x)$.*

The Behavior of a Rational Function Near $\pm\infty$

To determine the behavior of a rational function near ∞ or near $-\infty$, separately factor out the term with highest degree in the numerator and the denominator.

Example 6 *(A rational function.)*

Definition 2 *An* asymptote *of a graph is a line that the graph gets arbitrarily close to, in at least one direction along the line.*

Example 7 *(A rational function with lower-degree numerator than the denominator.)*

Example 8 *(A rational function with equal-degree numerator with the denominator.)*

Example 9 *(A rational function with higher-degree numerator than the denominator.)*

Graphs of Rational Functions

One major difference between the graph of a rational function and the graph of a polynomial is that a rational function may have vertical asymptotes.

Example 10 *(A rational function with vertical asymptote.)*