1.6 A Graphical Approach to Inverse Functions

The Graph of an Inverse Function

Consider the graph of the function $f(x) = x^2 + 1$, over the domain $[0, 3]$. This function is one-to-one and has an inverse function. The inverse function is $f^{-1}(x) = \sqrt{x-1}$. The graphs of a function and its inverse, drawn on the same set of axes, are reflections of each other around the line $y = x$.

This property is universal for all functions and their inverse functions. So for every point (a, b) that is on the graph of $f(x)$, the point (b, a) is on the graph of $f^{-1}(x)$. That is, we may draw the graph of $f^{-1}(x)$ by interchanging the numbers in the coordinate pairs of each point.

- The graph of a function and the graph of its inverse are symmetric with respect to the line $y = x$.
- Each graph can be obtained from the other by reflection throught the line $y = x$.

The above concept can help us draw graphs of inverse functions even if we do not have a formula for them. For example, if $f(x) = x^5 + x^3$, there is no formula for f^{-1} possible. However, we may reflect the graph of $f(x)$ around the line $y = x$ to obtain the graph of $f^{-1}(x)$. We do so by obtaining the reflection of key points and connecting them.

Inverse Functions via Tables

When a function f is given as table values, f is one-to-one when no number is repeated in the column labeled $f(x)$, thus making f invertible. We obtain the table for f^{-1} from the table of f by interchanging the two columns of input-output.

Interchanging the columns of a table of values for a one-to-one function f gives a table of values for the inverse function f^{-1}.

Graphical Interpretation of One-to-One

We may know whether a function is one-to-one just by looking at its graph, using the concept of *horizontal line test*.

A function is one-to-one if and only if every horizontal line intersects the graph of the function in at most one point.

So if we find even a single horizontal line that intersects the graph in more than one point, then the function is not invertible. For a function to be one-to-one, *every* horizontal line must intersect the graph in no more than one point.

Increasing and Decreasing Functions

Definition 1 *A function f is called **increasing** on an interval if $f(a) < f(b)$ whenever $a < b$ and a, b are in the interval. A function is called increasing if the function is increasing over its domain. A function f is called **decreasing***

on an interval if $f(a) > f(b)$ *whenever* $a < b$ *and* a, b *are in the interval. A function is called decreasing if the function is decreasing over its domain.*

Fact 1 *Every increasing function is one-to-one. Every decreasing function is one-to-one.*