1.5 Inverse Functions

Examples of Inverse Functions

Suppose that we have a function that adds 3 to its input:

$$f(x) = x + 3.$$

The question is can we guess the input if we knew the output? For instance, if the output is 13, what would be the input? How about when $f(x) = 29$, what is x? In general, if we know $y = f(x) = x + 3$, what is x?

The last question above demonstrates the concept of an inverse function. An inverse function, denoted by f^{-1}, is a function that undoes whatever the function f has done.

$$y = f(x) \Leftrightarrow x = f^{-1}(y).$$

Note: We read $f^{-1}(y)$ as "f inverse of y." This is a notation and the negative one is not an exponent: $f^{-1} \neq \frac{1}{f}$. In particular, if we want to show an exponent for the function f, we would write $[f(x)]^{-1} = \frac{1}{f(x)}$.

Not all functions have inverse function. For example, the function $g(x) = x^2 + 1$, defined over all real numbers, does not have an inverse. To see this, suppose we have $g(x) = 17$. The question is what is x to produce such a number? One answer is $x = 4 : 4^2 + 1 = 17$. However, this is not the only number; $x = -4$ also produces 17! So the reverse direction is not a function! However, we may fix this problem of lack of a unique solution to an inverse problem, by restricting the domain. Here, if the domain is nonnegative numbers, then we would get a unique x for each y.

$$y = x^2 + 1 \Leftrightarrow x = \sqrt{y - 1}, \quad x \geq 0, y \geq 1.$$

Definition 1. *A function f is called one-to-one if for each number y in the range of f, there is exactly one number x in the domain of f such that $f(x) = y$.*

For example, $g(x) = x^2 + 1$ over the real numbers is *not* one-to-one, but over the domain of nonnegative numbers *is* one-to-one.

Definition 2. *Suppose f is a one-to-one function and y is in the range of f. Then $f^{-1}(y)$ is defined to be the number x such that $f(x) = y$.*

So when f is a one-to-one function,

$$f(x) = y \Leftrightarrow f^{-1}(y) = x.$$

To find a formula for an inverse function to f, we solve the equation $f(x) = y$ for x in terms of y.

Example 1. *Suppose $h(x) = 3 - \frac{x+4}{x-7}$. Find $h^{-1}(y)$.*

The Domain and Range of an Inverse Function

If f is a one-to-one function, then

- the domain of f^{-1} equals the range of f;
- the range of f^{-1} equals the domain of f.

The Composition of a Function and Its Inverse

Since a function and its inverse undo each other (they are inverse of each other), we have

$$f(f^{-1}(y)) = y \quad \text{for every } y \text{ in the range of } f;$$
$$f^{-1}(f(x)) = x \quad \text{for every } x \text{ in the domain of } f.$$