7.5 The Area Between Two Curves

If f and g are continuous functions and $f(x) \geq g(x)$ on $[a, b]$, then the area between the curves $f(x)$ and $g(x)$ from $x = a$ to $x = b$ is

$$\int_a^b [f(x) - g(x)]\, dx.$$

7.5.1 Consumers' Surplus

Suppose you enter a store to buy an item for \$10. However, you discover that the item is actually selling for \$7. The amount that you saved (\$3), plus all the amount that other people save by buying at a lower amount that they were willing to pay, is called the consumer surplus.

Graphically, the consumer surplus is the area under the demand curve that is above the equilibrium:

$$\text{consumers' surplus} = \int_0^{q_0} [D(q) - p_0]\, dq,$$

where $p = D(q)$ is the demand function, and (q_0, p_0) is the equilibrium point.

Similarly, if manufacturers would be willing to supply a product at a price *lower* than the equilibrium price p_0, manufacturers would get added income, which is called the producers' surplus: the total of the differences between the equilibrium price and the lower prices at which the manufacturers would sell the product.

$$\text{producers' surplus} = \int_0^{q_0} [p_0 - S(q)]\, dq,$$

where (q_0, p_0) is the equilibrium point and $p = S(q)$ is the supply function.

The total surplus is the sum of consumers' surplus and producers' surplus.

Example 1. *A factory has installed a new process that will produce an increased rate of revenue (in thousands of dollars per year) of*

$$R'(t) = 104 - 0.4e^{t/2},$$

where t is time measured in years. The new process produces additional costs (in thousands of dollars per year) at the rate of

$$C'(t) = 0.3e^{t/2}.$$

a) When will it no longer be profitable to use this new process?

b) Find the net total savings.

Example 2. *Suppose the supply function for concrete is given (in dollars) by*

$$S(q) = 100 + eq^{3/2} + q^{5/2},$$

and that supply and demand are in equilibrium at $q = 9$. Find the producers' surplus.

Example 3. *Find the consumers' surplus if the demand function for extra virgin olive oil is given (in dollars) by*

$$D(q) = \frac{32000}{(2q + 8)^3},$$

and if supply and demand are in equilibrium at $q = 6$.

Example 4. *For a family with an income in the lowest fifth of the U.S. population, the mean household income for the years 1970 to 2012 can be approximated (in thousands of dollars) by*

$$f(t) = -0.0001844t^3 + 0.009938t^2 - 0.08515t + 11.33,$$

where t is the number of years since 1970. The mean household income for a family in the top 5% can be approximated by

$$g(t) = -0.01107t^3 + 0.6854t^2 - 6.587t + 193.3.$$

a) Find the area under $f(t)$ and $g(t)$, as well as the area between the two curves, over the interval $0 \leq t \leq 42$.

b) Interpret the answers from part (a) and discuss what they might tell us about income inequality.

Homework

§7.5: 27, 31, 33, 37, 41