7.4 The Fundamental Theorem of Calculus

We have seen that if $f(x) \geq 0$, then $\int_a^b f(x)\ dx$ is the area under $f(x)$ and above the x-axis, between $x = a$ and $x = b$.

The Fundamental Theorem of Calculus (FTOC) shows how differentiation and integration would undo each other.

Theorem (Fundamental Theorem of Calculus)**.** *Let f be a continuous function over the interval $[a, b]$, and let F be* any *antiderivative of f. Then*

$$\int_a^b f(x)\ dx = F(b) - F(a).$$

Note that $f(x)$ does not have to be nonnegative.

7.4.1 Properties of Definite Integrals

- $\int_a^b kf(x)\ dx = k\int_a^b f(x)\ dx$ for every constant k and function f.
- $\int_a^b (f+g)\ dx = \int_a^b f\ dx + \int_a^b g\ dx$ for functions f, g.
- $\int_a^a f(x)\ dx = 0$ for every function f
- $\int_a^b f(x)\ dx = \int_a^c f(x)\ dx + \int_c^b f(x)\ dx$ for every constant c
- $\int_b^a f(x)\ dx = -\int_a^b f(x)\ dx$

Since a definite integral gives a negative answer if the function is negative, the area for the regions that are below the x-axis is the opposite of the integral. Thus, to find the area bounded by a function and the x-axis, we find where the function crosses the x-axis to convert the negative answers to positive via absolute value, then add all the areas.

Example 1. *A worker new to a job will improve his efficiency with time so that it takes him fewer hours to produce an item with each day on the job, up to a certain point. Suppose the rate of change of the number of hours it takes a worker in a certain factory to produce the xth item is given by*

$$H'(t) = 20 - 2x.$$

a) What is the total number of hours required to produce the first 5 items?

b) What is the total number of hours required to produce the first 10 items?

Example 2. *An oil tanker is leaking oil at the rate of*

$$L'(t) = \frac{80\ln(t+1)}{t+1}$$

barrels per hour, where t is the time (hours) after the tanker hits a hidden rock (when $t = 0$).

a) Find the total number of barrels that the ship will leak on the first day.

b) Find the total number of barrels that the ship will leak on the second day.

c) What is happening over the long run to the amount of oil leaked per day?

Example 3. *Based on data from the U.S. Census Bureau, an approximate family income distribution for the United States is given by the function*

$$f(x) = 0.00826x^3 - 0.211x^2 + 0.74x + 9.36,$$

where x is the annual income in units of \$10,000, with $0 \leq x \leq 10$. For example, $x = 0.5$ represents an annual family income of \$5000. The percent of families with an income in a given range can be found by integrating this function over that range. Find the percentage of families with an income between \$35,000 and \$60,000.

Example 4. *The rate of consumption of oil (in billions of barrels) by a company is given as*

$$1.2e^{0.04t},$$

where $t = 0$ corresponds to 2010. Find the total of oil used by the company from 2010 to year T. At this rate, how much will be used in 5 years?

Homework

§7.4: 55, 59, 69, 71