5.3 Higher Derivatives

We have seen how derivative is the rate of change of a function. The derivative itself is a function. Thus the rate of change of the derivative tells us how the derivative is changing. The derivative of the first derivative is called the second derivative. For a function $y = f(x)$, we show the second derivative by either $f''(x)$ (read "f double prime of x"), or d^y/dx^2. Higher derivative are also possible. For instance, the third derivative is the rate of change of the second derivative.

$$\begin{aligned} y &= f(x) \\ \frac{dy}{dx} &= f'(x) \\ \frac{d^2y}{dx^2} &= f''(x) \\ \frac{d^3y}{dx^3} &= f'''(x) \\ \frac{d^4y}{dx^4} &= f^{(4)}(x) \\ &\vdots \\ \frac{d^ny}{dx^n} &= f^{(n)}(x) \\ &\vdots \end{aligned}$$

Note that after the third derivative, we write the derivative order in parentheses rather than using the prime notation. We have seen second derivatives before: the acceleration is the second derivative of position. If $a(t)$ is the acceleration at time t, v is the velocity, and x is the position, then:

$$a(t) = \frac{dv}{dt} = \frac{d^2x}{dt^2}.$$

We may use the second derivative to find out where a curve is curved up (called concave up) or down (concave down). For a concave up curve, the first derivative is increasing, thus the second derivative is positive. Similarly, when a curve is concave down, the second derivative is negative. A point where concavity changes is called an inflection point or point of diminishing returns.

$$\begin{aligned} f'' > 0 &\iff \text{concave up} \\ f'' < 0 &\iff \text{concave down} \end{aligned}$$

$$\text{inflection point at } x = c \Longrightarrow f'' = 0 \text{ or undefined.}$$

Again note that the direction is one-way only: if $f'' = 0$ we may or may not have an inflection point.

The second derivative test is a way to test a critical point using the second derivative, because at maximum we have concave down and at minimum we have a concave up curve. Suppose c is a critical point ($f'(c) = 0$ or undefined). Then

$$\begin{aligned} f''(c) > 0 &\Longrightarrow f(c) \text{ is a minimum} \\ f''(c) < 0 &\Longrightarrow f(c) \text{ is a maximum} \\ f''(c) = 0 &\Longrightarrow \text{no conclusion.} \end{aligned}$$

Example 1. *Suppose the revenue, in thousands of dollars, from spending x thousands of dollars on advertising, is*

$$R(x) = -0.3x^3 + x^2 + 11.4x, \quad 0 \le x \le 6.$$

Find the point of diminishing returns.

Example 2. *The population of whooping cranes after t years is*

$$P(t) = \frac{787}{1 + 60.8e^{-0.0473t}}.$$

Find the inflection point.

Example 3. *A car rolls down a hill. Its distance, in feet, from its staring point is*

$$x(t) = 1.5t^2 + 4t,$$

where t is in seconds.

a) How far will the car move in 10 seconds?

b) What is the velocity $v(t)$ at 10 seconds?

c) How can you tell from $v(t)$ that the car will not stop?

d) What is the acceleration $a(t)$ at 10 seconds?

e) What is happening to $v(t)$ and $a(t)$ as t increases?

Homework

§5.3: 75, 79, 85, 93