5.2 Relative Extrema

Optimization is an important topic in many fields. To optimize a function means to find the optimum value of the function. The optimum value, that is, the best value, is often the maximum (such as maximum profit) or minimum (such as minimum cost).

We may find the extreme values of a function with the help of derivatives. The maximum of a function is the peak of the graph and the minimum is the bottom of a valley.

A relative, or local, maximum is a point higher than the nearby points. Mathematically,

$$f(c) \text{ is a local maximum} \Longleftrightarrow f(c) \geq f(x) \text{ for all } x \text{ near } c$$

Similarly,

$$f(c) \text{ is a local minimum} \Longleftrightarrow f(c) \leq f(x) \text{ for all } x \text{ near } c$$

We notice a characteristic of every extreme point on a graph: the derivative is either zero or undefined there. We call those points at the derivative is zero or undefined, a critical point. Note that a critical point is a candidate for an extreme value. Not all critical points are extreme values.

$$\text{extreme value} \Longrightarrow \text{critical point}$$

However, the converse is not necessarily true:

$$\text{extreme value} \not\Longleftarrow \text{critical point}$$

A counterexample is the function $f(x) = x^3$.

A procedure for finding extreme points is to find critical points. Then check the critical points to see whether any of them is an extreme point.

The first derivative test is a method to test critical points by looking at the first derivative:

Suppose $f'(c) = 0$ or $f'(c)$ is undefined. Then we look at the $f'(x)$ on either side of $x = c$:

$$[f'(x) > 0 \text{ for } x < c] \text{ and } [f'(x) < 0 \text{ for } x > c] \Longrightarrow f(c) \text{ is a maximum}$$

$$[f'(x) < 0 \text{ for } x < c] \text{ and } [f'(x) > 0 \text{ for } x > c] \Longrightarrow f(c) \text{ is a minimum}$$

Example 1. *Suppose the cost of producing q items is $C = 25q + 5000$ dollars and the price of each item sold is $p = 90 - 0.02q$ dollars.*

a) Find the number of units q that produces maximum profit.

b) Find the price per unit p that produces maximum profit.

c) Find the maximum profit.

Example 2. *The average individual daily milk consumption for herds of calves can be modeled by*

$$M(t) = 6.281t^{0.242}e^{-0.025t}, \qquad {\scriptstyle 1 \leq t \leq 26,}$$

where $M(t)$ is the milk consumption in kilograms and t is the age of the calf in weeks.

a) Find the time in which the maximum daily consumption occurs and the maximum daily consumption.

b) If the general formula for this model is

$$M(t) = at^b e^{-ct},$$

find the time when the maximum consumption occurs and the maximum consumption.

Example 3. *A group of researchers found that people prefer training films of moderate length; shorter films contain too little information, while longer films are boring. For a training film on the care of exotic birds, the researchers determined that the ratings people gave for the film could be approximated by*

$$R(t) = \frac{20t}{t^2 + 100},$$

where t is the length of the film in minutes. Find the film length that received the highest rating.

Homework

§5.2: 43, 51, 53, 57, 59