4.4 Derivatives of Exponential Functions

The most basic exponential function in calculus has base e:

$$y = f(x) = e^x.$$

The exponential function with base e, is the only function (other than the trivial zero function), whose rate of change is itself:

$$\frac{d}{dx}\left(e^x\right) = e^x.$$

If the base is not e, we may find the derivative as follows.

$$\begin{aligned}
\frac{d}{dx}\left(b^x\right) &= \frac{d}{dx}\left(e^{\ln b^x}\right) \\
&= \frac{d}{dx}\left(e^{x\ln b}\right) \\
&= e^{x\ln b}\cdot\frac{d}{dx}\left(x\ln b\right) \qquad \text{by Chain Rule} \\
&= e^{x\ln b}\cdot\ln b \\
&= (\ln b)b^x.
\end{aligned}$$

Example 1. *After the introduction of a new product for tanning without sun exposure, the percent of the public that is aware of the product is approximated by*

$$A = 10t^2 2^{-t},$$

where t is the time in months. Find the rate of change of the percent of the public that is aware of the product after the following number of months

a) 2

b) 4

c) Notice that the answer to (a) is positive and the answer to (b) is negative. What does this tell you about how public awareness of the product has changed?

When population begin growing, they usually grow exponentially. However, a population may not grow exponentially forever (why?). A model that fits what we observe more realistically is the logistic model:

$$P = \frac{M}{1 + \left(\frac{M}{P_0} - 1\right)e^{-kMt}},$$

where P is the population, t is time, P_0 is the initial population, M is maximum possible size of the population, and k is a positive constant. Note that as $t \to \infty$, $e^{-kMt} \to 0$ and hence $P \to M$.

Example 2. *Based on data from the U.S. Fish and Wildlife Services, the population of whooping cranes in the Aransas-Wood Buffalo National Park can bre approximated by a logistic function with $k = 6.01 \times 10^{-5}$, with a population in 1958 of 32 and a maximum population of 787.*

a) Find the growth function $P(t)$ for the whooping crane population, where t is time since 1938, when the park first started counting the cranes.

b) Find the initial population P_0.

c) Find the population rate of growth in the following years:

i) 1945

ii) 1985

iii) 2005

d) What happens to the rate of growth over time?

Example 3. *The age/weight relationship of female Arctic foxes caught in Svalbard, Norway, can be estimated by*

$$M = 3102e^{-e^{0.022(t-56)}},$$

where t is the age of the fox in days and M is the weight of the fox in grams.

a) Estimate the weight of a female fox that is 200 days old.

b) Use the model to estimate the largest size that a female fox can attain.

c) Estimate the age of a fox when it has reached 80% of its maximum size.

d) Estimate the rate of change in weight of an Arctic fox that is 200 days old.

Example 4. *The amount (in grams) of a sample of lead 214 present after t years is*

$$A = 500e^{-0.31t}.$$

a) Find the rate of change of the quantity present after 6 years.

b) Show that the rate of change is always negative. What does this mean?

c) What is happening to the rate of change of the amount present as the number of years increases?

d) Will the substance ever be gone completely?

Homework

§4.4: 39, 47, 55