4.3 The Chain Rule

The chain rule is a rule for differentiating more complicated functions, that is, functions of functions. These functions are called composite functions. Each composite function is composed of simpler functions.

4.3.1 Composition of Functions

A composition of functions f and g may look like the following:

$$g(f(x)),$$

read "g of f of x." In $g[f(x)]$, the input x goes to the function f and the output of the function f becomes the input for function g. So, x is in the domain of f and $y = f(x)$ is in the domain of g.

4.3.2 Chain Rule

Now, consider the following. Suppose person X works twice as fast as person Y and person Y works 3 times as fast as person Z. How many times faster is person X working compared to person Z?

The above example demonstrates the concept behind the chain rule. Speed is an example of a rate of change and rate of change is the derivative. We may write the chain rule as follows:

$$\frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx}$$

The above statement says to obtain the rate of change of y with respect to x, we may multiply the rate of change of y with respect to another variable z, with the rate of change of that z with respect to x. The formula also makes sense if we look how dz cancels on the right side to obtain the left side.

Another way to write the chain rule is as follows:

$$\frac{d}{dx}[f(g(x))] = f'(g) \cdot g'(x).$$

Example 1. *Assume the total revenue (in dollars) from the sale of x TV sets is given by*

$$R = 24(x^2 + x)^{2/3}.$$

a) Find the marginal revenue function.

b) Find the average revenue from the sale of x sets.

c) Find the marginal average revenue.

d) How does the revenue change over time? What does the marginal revenue function tell you about the revenue function? What does the average revenue function tell you about the revenue function?

Example 2. *Suppose the cost in dollars of manufacturing q items is*

$$C = 2000q + 3500$$

and the demand equation is

$$q = \sqrt{15000 - 1.5p}.$$

In terms of the demand q,

a) find an expression for revenue R.

b) find an expression for profit P.

c) find an expression for marginal profit.

d) determine the value of marginal profit when the price is $5000.

e) based on the answer to the previous part, should we increase the price or decrease the price for more profit?

The total number of bacteria (in millions) present in a culture is

$$N = 2t(5t + 9)^{1/2} + 12,$$

where t represents time (in hours) after the beginning of an experiment. Find the rate of change of the population of bacteria with respect to time for the following number of hours:

a) 0

b) 8

Example 3. *Zenzizenzizenzic is an obsolete word with the distinction of containing the most z's of any word found in the Oxford English Dictionary. It was used in mathematics, before powers were written as superscript numbers, to represent the square of the square of the square of a number. In symbols, zenzizenzizenzic is written as* $(((x^2)^2)^2$.

a) Use the chain rule twice to find the derivative.

b) Use the properties of exponents to first simplify the expression, and then find the derivative.

Homework

§4.3: 53, 55, 59, 67