3.4 Definition of the Derivative

3.4.1 The Tangent Line

Recall that we used the formula

$$\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$$

to calculate the (instantaneous) rate of change of a function f at $x=a$. Geometrically, this limit means that the slope of a secant line approaches the slope of the tangent line at a point where $x=a$. A secant line is a straight line that connects two points on a curve.

The tangent line of the graph of $y=f(x)$ at the point $(a,f(a))$ is the line through this point having slope

$$\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$$

provided this limit exists. If this limit does not exist, then there is no tangent at the point.

Note that $h=\Delta x$ means the change in x.

3.4.2 The Derivative

The derivative of the function f at x is defined as

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{z\to x}\frac{f(z)-f(x)}{z-x}$$

provided this limit exists.

Note that $f'(x)$ (read "f prime of x") is itself a function of x. The function $f'(x)$ is called the derivative of f with respect to x. If $f'(x)$ exists, we say f is differentiable. at x. The process that produces f' is called differentiation.

Compare the following between difference quotient and the limit of the difference quotient:

Difference Quotient	Derivative
$\frac{f(x+h)-f(x)}{h}$	$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$
slope of a secant line	slope of the tangent line
average rate of change	instantaneous rate of change
average velocity	instantaneous velocity
average rate of change in cost	marginal cost
average rate of change in revenue	marginal revenue
average rate of change in profit	marginal profit

Example 1. *The profit, in thousands of dollars, from the expenditure of x thousands of dollars on advertising is*

$$P(x)=1000+32x-2x^2.$$

Find the marginal profit at the following expenditures. In each case, decide whether the firm should increase the expenditure.

a) \$8000

b) \$6000

c) \$12000

The tangent line to the graph of $y=f(x)$ at the point $(x_0,f(x_0))$ is given by the equation

$$f-f(x_0)=f'(x_0)(x-x_0),$$

provided $f'(x)$ exists.

Example 2. *The cost in dollars of producing x tacos is*

$$C(x) = -0.00375x^2 + 1.5x + 1000, \quad 0 \le x \le 180.$$

a) Find the marginal cost.

b) Find and interpret the marginal cost at a production level of 100 tacos.

c) Find the exact cost to produce the 101st taco.

d) Compare the answers to parts (b) and (c). How are they related?

Example 3. *Suppose we graph the temperature T, in $^\circ F$, as a function of time t, in hours. We may estimate $T'(3)$ by estimating the slope of the graph at $t = 3$. If the slope is positive, it means the temperature is rising. If the slope is negative, it means the temperature is dropping. If the slope is zero, it means the temperature is steady.*

The derivative exists when a function f satisfies *all* of the following conditions at a point.

- f is continuous
- f is smooth, and
- f does not have a vertical tangent line.

If any of the above conditions is not true, then the derivative does *not* exist at that point. So, if f is discontinuous, or if f has a sharp corner, or if f has a vertical tangent line, then f is not differentiable.

Homework

§3.4: 49, 51, 55