3 The Derivative

3.1 Limits

Definition. *Suppose f is a function, a is a number, and L is a number. We say the limit of f as x approaches a is L:*

$$\lim_{x\to a} f(x) = L$$

if as x gets closer and closer to a (and not equal to a) on both sides of a, $y = f(x)$ gets closer and closer to L. This means we can make f as close to L as we like, provided we make x close enough to a.

The notation for one-sided limits are as follows:

$\lim_{x\to a^-} f$ is the limit of f as x approaches a from the left of a towards a.

$\lim_{x\to a^+} f$ is the limit of f as x approaches a from the right of a towards a.

It follows that $\lim_{x\to a} f$ exists if and only if the left and right limits equal each other. In other words, if $\lim_{x\to a^-} f \neq \lim_{x\to a^+} f$, then $\lim_{x\to a} f$ does not exist.

Furthermore, if $\lim_{x\to a} f = L$ and L is not a number, then $\lim_{x\to a} f$ does not exist. In particular, if $L = \infty$ or if $L = -\infty$, then we say the limit of f does not exist.

There are certain rules for limits, such as the following limits as $x \to a$, as long as the limits exist, for functions f, g and constant $b > 0, b \neq 1$:

$$\lim(fg) = \lim(f) \cdot \lim(g)$$
$$\lim(b^{f(x)}) = b^{\lim f(x)}$$
$$\lim[\log_b f(x)] = \log_b[\lim f(x)]$$

Both x and $y = f(x)$ may go towards infinity or negative infinity. Note that infinity is not a number, rather a concept. Infinity is larger than every number. To say that $x \to \infty$ means x gets larger and larger without bound. To say that $x \to -\infty$ means that x gets smaller and smaller without bound.

Example 1.

$$\lim_{x\to\pm\infty} \frac{1}{x^n} = 0$$

For polynomials, it is easy to calculate limit: just plug in. Suppose $p(x)$ is a polynomial:

$$p(x) = a_n x^n + a_{n-1}x^{n-1} = \cdots + a_1 x + a_0$$

Then

$$\lim_{x\to a} p(x) = p(a)$$

Example 2. *The following picture shows $f(x)$. Find the following:*

a) $\lim_{x\to 2^-} f$

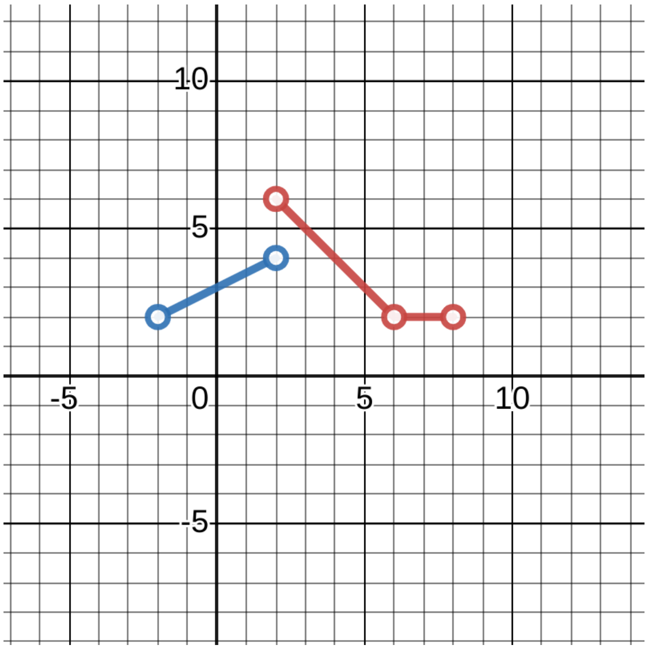
b) $\lim_{x\to 2^+} f$

c) $\lim_{x\to 2} f$

d) $\lim_{x\to 6^-} f$

e) $\lim_{x\to 6^+} f$

f) $\lim_{x\to 6} f$



Example 3. *The cost to fly* x *miles on American Airlines is* $C(x) = 0.0417x + 167.55$ *dollars. The average cost, denoted by* $\bar{C}(x)$*, is* $\bar{C}(x) = C/x$*. Find and interpret* $\lim_{x\to\infty} \bar{C}(x)$*.*

Example 4. *The number of teeth* N *at time* t *days of incubation for alligator is modeled by*

$$N(t) = 71.8e^{-8.96e^{-0.0685t}}$$

1. *Find* $N(65)$*, the number of teeth of an alligator that hatched after 65 days.*
2. *Find* $\lim_{t\to\infty} N(t)$ *and use this value as an estimate of the number of teeth of a newborn alligator.*

Example 5. *The concentration of a drug in a patient's bloddstream* h *hours after it was injected is given by*

$$A(h) = \frac{0.17h}{h^2 + 2}.$$

Find and interpret $\lim_{h\to\infty} A(h)$*.*

Homework

§3.1: 85, 87, 93