2 Nonlinear Functions

2.1 Properties of Functions

Think of a vending machine. If we select Pepsi and sometimes get Pepsi and sometimes Coke, then the machine is not functioning. For a function, each input must to a single output, not more than one output. Usually we assign the variable x for inputs and the variable y for outputs and write y as a function of x: $y = f(x)$. The set of inputs is called the domain and outputs are from a set called the range.

Graphically, since for every x there should be only one y, if we draw a vertical line, that vertical line could not cross a function more than once. If we can find a vertical line that crosses a graph more than once, that graph cannot be a function. This is called the vertical line test.

Example 1. *(2.1.72) Suppose $C(t)$ is the energy consumption of China for year t from year 1990 to the year 2035.*

a) What is the domain of the function?

b) Estimate the range of the function from the graph (suppose from lowest of about 25 quadrillion Btu in 2000 to about 200 quadrillion Btu in 2035).

2.2 Quadratic Functions

Every function $f(x)$ of the form $f(x) = ax^2+bx+c$, where a, b, c are constants, is called a quadratic function. Note the highest power of the input x is 2, hence the name quadratic. The constant $a \neq 0$, otherwise there is no x^2 term and the function would become linear.

2.2.1 Graph of Quadratic Function

The graph of a quadratic function is called a parabola. Every parabola is symmetric around an axis of symmetry that passes through the turning point, called the vertex. If $f(x) = ax^2 + bx + c$, then the vertex is at $x = -\frac{b}{2a}$. We find the x-intercepts when $y = f(x) = 0$. The quadratic formula gives us the x-intercepts, if they exist:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

If the number under square root is negative, then there is no x-intercept.

Example 2. *(2.2.50) Suppose the cost of producing x batches of widgets is $C(x) = \frac{3}{2}x + 3$ thousands of dollars. Furthermore, the revenue from selling x batches is $R(x) = -\frac{x^2}{2} + 5x$ thousands of dollars.*

a) Graph both functions.

b) Find the minimum break-even quantity (minimum = lowest, smallest).

c) Find the maximum revenue (maximum = highest, most).

d) Find the maximum profit.

2.4 Exponential Functions

An exponential function with base b is of the form $f(x) = b^x$ where $b > 0$ and $b \neq 1$. (Why?) We have two kinds of exponential functions: exponential growth and exponential decay. It's useful to know the properties of exponents, such as b^0.

2.4.1 Compound Interest

When we borrow money, we usually have to pay back more than we borrowed. The extra amount is called the interest. Similarly, when we lend our money to a bank, the bank gives us our money back, plus extra money called the interest. The original amount is called the principal, usually denoted by the variable P. The variable r is used for the interest rate (for a year). And t represents time in years.

The simple interest, I, is

$$I = Prt$$

Now if we lend our money and do not collect the interest, the interest is added to the principal. Then next year the new principal, which is the original principal plus the interest, collects interest. Thus we will collect interest on top of interest, that is, the previous interest collects interest itself. This is called compound interest. After the first year, we have

$$P + Pr(1) = P + Pr = P(1 + r)$$

After the second year, we have

$$P(1 + r) + P(1 + r)r = P(1 + r)(1 + r) = P(1 + r)^2$$

Similarly, after the third year, the total amount will be $P(1 + r)^3$. Notice the pattern: after the t years, the total amount will be $P(1 + r)^t$. Thus the total amount is an exponential function of $(1 + r)$.

Now if we collect the interest n times each year, for each period, the interest rate is r/n. Thus the total amount after t years will be

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

Let's consider an extreme example.

Example 3. *Suppose we invest a principal of \$1 with interest rate of 100%. Calculate the total amount after one year, as we increase the number of times we collect interest during that year. Does the total amount increase without bound?*

The previous example results in one of the five fundamental numbers in all mathematics. This number is so famous that we use a single letter to denote it.

$$e = \lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n$$

Because of the phenomenon we observe in the previous example, we obtain the formula for continuous compounded interest:

$$A = Pe^{rt}$$

Example 4. *(2.4.38) Suppose \$26000 is borrowed for 4 years at 6% interest. Find the interest paid over this period if the interest is compounded as follows:*

a) quarterly

b) monthly

c) continuously

Example 5. *(2.4.44) Lauren puts \$10500 into an account to save money to buy a car in 12 years. She expects the car of her dreams to cost \$35000 by then. Find the interest rate that is necessary if the interest is compounded monthly.*

Example 6. *(2.4.49) Suppose we use the model for world population P, in millions, at time t years since 1960, by the formula*

$$P(t) = 3100e^{0.0166t}$$

World population was about 3686 million in 1970. How closely does the model estimate this value?

2.5 Logarithmic Functions

Consider the following table. How could we fill the following table correctly?

x	2	8	4	16	1	1/2	64	
y								5

How about the following table?

x	1	2	3	4	5	6	7	8	9	10
y		0.301	0.478							

The inverse function for the exponential function is called logarithm. Thus we have the following equivalent equations:

$$x = b^y \iff y = \log_b x$$

In other words, logarithm is really the exponent.

Remark. *Logarithm is the name of a function, so* log *by itself has no meaning. The convention is to drop the parentheses from the function name for logarithm, so* $\log(x)$ *is written as* $\log x$ *when there is no ambiguity.*

Remark. *If $x = b^y$, then since $b > 0$, we must have $x > 0$. Thus when $y = \log_b x$, we must have $x > 0$. That is, the input for logarithmic function must be strictly positive (not zero and not negative).*

Since logarithm is the exponent, logarithm has the same properties of exponents. In particular:

$$\log_b(xy) = \log_b(x) + \log_b(y)$$
$$\log_b\left(\frac{x}{y}\right) = \log_b x - \log_b y$$
$$\log_b(x)^n = n \log_b x$$

A few particular values of logarithms:

$$\log_b(1) = ?$$
$$\log_b(b) = ?$$

Remark. *Logarithm with base e may be written as* ln*, called the natural logarithm. For example,* $\ln e = 1$ *and* $\ln(e)^3 = 3$.

Remark. *Another formula that is sometimes useful is change of base:*

$$\log_b(a) = \frac{\log_c(a)}{\log_c(b)}$$

The change of base formula is useful when we want to calculate the value of a logarithm with a calculator that only has base 10 and base e. Base 10 logarithms are usually written without the base. For example, $\log 10 = 1$ *and* $\log 1000 = 3$.

As a general rule, when we solve an exponential equation, where the unknown is in the exponent, we use the inverse of exponential function, that is, the logarithm, to solve. On the other hand, when we have logarithmic equation, when the unknown is inside a logarithm, we use the inverse of logarithm, that is, exponential function, to solve.

Remark. *We may change the base of every exponential function to base e by using the inverse functions:*

$$b^x = e^{\ln(b^x)} = e^{x \ln b}$$

This conversion is especially useful when we want to calculate derivatives later.

Example 7. *(2.5.76) Alison invests \$15000 in an account paying 7% compounded annually. How many years are required for the compound amount to at least double? Note that interest is only paid at the end of each year.*

Example 8. *(2.5.92) The loudness of sounds is measured in a unit called a decibel. To do this, a very faint sound, called the threshold sound, is assigned an intensity I_0. If a particular sound has intensity I, then the decibel rating of this louder sound is*

$$10 \log \left(\frac{I}{I_0} \right)$$

Find the decibel ratings of the following sounds having intensities as given. Round answers to the nearest whole number.

a) Whisper, $115I_0$

b) Busy street, $9{,}500{,}000I_0$

c) Jet liner at takeoff, $109{,}000{,}000{,}000{,}000I_0$

Homework

§2.1: 71
§2.2: 49
§2.4: 37, 43, 52
§2.5: 77, 93