13.1 Definitions of the Trigonometric Function

Trigonometric functions are ideal to model repetitive (or periodic) phenomena, such as high tide and low tide, and temperature.

The input for a trigonometric function is an angle and the output is a number. Thus we begin with angles.

A ray is a portion of a line that has one endpoint. If we rotate the ray around the endpoint, we obtain an angle between the initial side and the terminal side of the angle. The endpoint of the ray is called the vertex of the angle. We say an angle is in standard position when the vertex is on the origin and the initial side is along the positive x-axis. We say the angle is positive if we rotate counterclockwise from the initial side to the terminal side. The angle is negative if we rotate clockwise.

13.1.1 Degree Measure

More than 4000 years ago the Babylonians divided the circle into 360 equal parts, each called a degree. A few examples are right angle at $90°$, a straight line at $180°$, and a full circle at $360°$.

13.1.2 Radian Measure

One radian is the measure of a central angle in a circle such that the length of the opposite are is the same as the radius. Alternatively,

$$\text{Radian measure of } \theta = \frac{\text{Length of arc}}{\text{Length of radius}} = \frac{s}{r}$$

Notice that the definition of radian has two lengths divided. Thus we may think of radian as a number without units. If the angle is in degrees, we write a little circle on top-right corner. Otherwise, if the angle is in radians, we do not have to write anything.

Since the circumference of a circle with radius r is $2\pi r$, a full circle is 2π radians. This gives us a way to convert between radians and degrees: $360° = 2\pi$. We may simplify this by dividing both sides by two:

$$180° = \pi$$

Example 1. *How many radians is* $30°$*?*

Example 2. *How many degrees is* $\frac{\pi}{4}$*?*

13.1.3 Trigonometric Functions

Consider a circle centered at origin with radius r. The radius from the origin to a point (x, y) on the circle forms an angle θ in standard position. By convention, we may drop the parentheses in the name of trigonometric functions (similar to logarithms). We define the six trigonometric functions as follows:

$$\begin{aligned} \sin\theta &= \frac{y}{r}, && \text{sine of angle theta} \\ \cos\theta &= \frac{x}{r}, && \text{cosine of angle theta} \\ \tan\theta &= \frac{y}{x}, && \text{tangent of angle theta} \end{aligned}$$

The other three trigonometric functions are reciprocals of the previous three:

$$\begin{aligned} \csc\theta &= \frac{1}{\sin\theta}, && \text{cosecant of angle theta} \\ \sec\theta &= \frac{1}{\cos\theta}, && \text{secant of angle theta} \\ \cot\theta &= \frac{1}{\tan\theta}, && \text{cotangent of angle theta} \end{aligned}$$

Note that the equation of the circle is based on Pythagorean Theorem:

$$x^2 + y^2 = r^2$$

If we divide both sides by r^2, we obtain the most important trigonometric identity:

$$\cos^2\theta + \sin^2\theta = 1$$

The convention is to put the exponent between the trigonometric function name and the input. So, for example, $\sin^2\theta$ is the same as $[\sin(\theta)]^2$.

Notice also that $\tan\theta = \frac{\sin\theta}{\cos\theta}$.

We may also define trigonometric functions on a right triangle with an acute angle θ:

$$\begin{aligned} \sin\theta &= \frac{opposite}{hypotenuse} \\ \cos\theta &= \frac{adjacent}{hypotenuse} \\ \tan\theta &= \frac{opposite}{adjacent} \end{aligned}$$

A mnemonic SohCahToa may remind us of these equations. This leads us to two special right triangles: one with angles 30° and 60°, the other with angles 45°.

For an arbitrary angle, we generally use a calculator to find the trigonometric function value. We have to pay attention whether the angle is in radian or in degrees.

We may find the values of the main three trigonometric functions at the highest/lowest values of x and y on the circle, which correspond to angles $0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$. Note that after 2π, all trigonometric functions repeat. We say the trigonometric functions are periodic. The period of a periodic function is the smallest input for which the function repeats. For sine and cosine, the period is 2π. Tangent, however, is the same as the slope of the radius. Since the angle of a straight line is π, the period of tangent is π, not 2π. Thus:

$$\begin{aligned} \sin(\theta + 2\pi) &= \sin\theta \\ \cos(\theta + 2\pi) &= \cos\theta \\ \tan(\theta + \pi) &= \tan\theta \end{aligned}$$

To draw a sketch of trigonometric functions, it is enough to find the values of 4 points in a period. After that, we repeat the sketch. Sine and cosine especially are suitable for modeling periodic phenomena. They both look like waves. The following functions are called sinusoidal:

$$y = a\sin(bt - c) + d, \qquad y = a\cos(bt - c) + d$$

where

$$\begin{aligned} |a| &= \text{amplitude} \\ \frac{2\pi}{b} &= \text{period} \\ \frac{1}{\text{period}} &= \text{frequency} \\ \frac{c}{b} &= \text{phase shift} \\ d &= \text{vertical shift} \end{aligned}$$

Example 3. *A mathematical model for the temperature in Fairbanks is*

$$T = 37\sin\left[\frac{2\pi}{365}(t - 101)\right] + 25,$$

where T is the temperature in $^\circ F$ on day t, with $t = 0$ corresponding to January 1 and $t = 364$ corresponding to December 31. What are the amplitude, period, frequency, phase shift, and vertical shift? Use a calculator to estimate the temperature on May 1 (day 121).

Example 4. *Jenny stands on a cliff at the edge of a canyon. On the opposite side of the canyon is another cliff equal in height to the one she is on. By dropping a rock and timing its fall, Jenny determines that it is 105 ft to the bottom of the canyon. She also determines the angle between horizontal and to the base of the opposite cliff is 27°. How far is it to the opposite side of the canyon?*

Homework

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