9 Orbits, Cycles, and the Alternating Groups

9.1 Orbits

Consider the equivalence relation $\sim$ for a permutation σ of a set A: $\forall a, b, \in A$,

$$a \sim b \iff \exists\, n \in \mathbb{Z} \text{ such that } b = \sigma^n(a)$$

It is straightforward to show that $\sim$ is reflexive, symmetric, and transitive, and hence, an equivalence relation.

Definition. *The equivalence classes in A determined by $\sim$ are **orbits** of σ.*

Example 1. *The orbits of the identity permutation ι:*

$$\forall a \in A, \mathcal{O}_{a,\iota} = \{a\}$$

Example 2. *Find all orbits of $\sigma : \mathbb{Z} \to \mathbb{Z}$ where $\sigma(n) = n + 1$.*

9.2 Cycles

Definition. *A permutation $\sigma \in S_n$ is a **cycle** if it has at most one orbit containing more than one element. The **length** of a cycle is the number of elements in its largest orbit. The identity permutation is a cycle of length 1.*

Example 3. *A cycle $(4, 8, 6)$ on a set $\{1, 2, 3, 4, 5, 6, 7, 8\}$ is*

$$(4, 8, 6) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 2 & 3 & 8 & 5 & 4 & 7 & 6 \end{pmatrix}$$

Cycles are permutations, so we may multiply them. The product of two cycles may or may not be a cycle.

Example 4. *Compute the product of cycles*

$$(1, 3, 2, 7)(4, 8, 6)$$

that are permutations of $\{1, 2, 3, 4, 5, 6, 7, 8\}$

In **disjoint** cycles no number appears in the notations of two different cycles.

Theorem. *Every permutation of a finite set is a product of disjoint cycles.*

Remark. *Permutation multiplication is not commutative. However, multiplication of disjoint cycles is commutative.*

9.3 Even and Odd Permutations

Definition. *A **transposition** is a cycle of length 2.*

Remark. *A transposition leaves all elements fixed, except two elements.*

Remark. *Every cycle is a product of transpositions:*

$$(a_1, a_2, \ldots, a_n) = (a_1, a_n)(a_1, a_{n-1}) \cdots (a_1, a_3)(a_1, a_2)$$

The following is a corollary to our previous theorem.

Corollary. *Every permutation of a finite set of at least two elements is a product of transpositions.*

Remark. *The number of transpositions to represent a permutation is either always even or always odd.*

Example 5. *The identity permutation is even. For example, we may write $\iota = (1,3)(3,1)$. In S_1, we define ι to be even.*

Theorem. *No permutation in S_n can be expressed both as a product of an even number of transpositions and as a product of an odd number of transpositions.*

Definition. *An even permutation is a product of an even number of transpositions. An odd permutation is a product of an odd number of transpositions.*

Example 6. *Express the following permutation of $\{1,2,3,4,5,6,7,8\}$ as a product of disjoint cycles, and then as a product of transpositions.*

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 8 & 2 & 6 & 3 & 7 & 4 & 5 & 1 \end{pmatrix}$$

Example 7. *Find the maximum possible order of an element of S_{15}.*

9.4 The Alternating Groups

Theorem. *If $n \geq 2$, then the collection of all even permutations of $\{1, 2, 3, \ldots, n\}$ forms a subgroup of order $n!/2$ of the symmetric group S_n.*

Definition. *The subgroup of S_n consisting of the even permutations of n letters is the **alternating group** A_n on n letters.*

Cayley's theorem shows that every finite group G is isomorphic to some subgroup of S_n for $n = |G|$.

Because of the structure of A_n, there are no general formulas involving only radicals for solutions of polynomial equations of degree $n \geq 5$.

Example 8. *Let σ be a permutation of a set A. We shall say "σ moves $a \in A$" if $\sigma(a) \neq a$. If A is a finite set, how many elements are moved by a cycle $\sigma \in S_A$ of length n?*