4 Groups

4.1 Definition and Examples

Example 1. *Solve:*

a) $-7 + x = -4$.

b) $-6x = -1$.

We choose our group properties to allow us to solve such equations.

Definition 1. *A* ***group*** $\langle G, * \rangle$ *is a set* G*, such that the following axioms are satisfied:*

- ***closure under*** $*$*:* $\forall a, b \in G, a * b \in G$.
- ***associativity of*** $*$*:* $\forall a, b, c \in G,\ (a * b) * c = a * (b * c)$.
- ***identity element*** e *for* $*$*:* $\exists e \in G$ *such that* $\forall x \in G,\ e * x = x * e = x$.
- ***inverse*** a' *of* a*:* $\forall a \in G,\ \exists a' \in G$ *such that* $a * a' = a' * a = e$.

Note that we may refer to the group $\langle G, * \rangle$ as the group G under $*$. If the operation is obvious, we may refer to the group by naming the set only. For example, we may say the group $\mathbb{Z}_7$.

$$\langle G, * \rangle \text{ is } \textbf{abelian} \iff \forall x, y \in G, x * y = y * x$$

Example 2. *Is* $\langle \mathbb{R}, \cdot \rangle$ *a group?*

Example 3. *The set of all invertible* $n \times n$ *matrices under matrix multiplication,* $\langle M_n(\mathbb{R}), \cdot \rangle$*, is a nonabelian group. In linear algebra, this group is called the general linear group of degree* n*, denoted by* $GL(n, \mathbb{R})$*.*

4.2 Elementary Properties of Groups

Theorem. *If* $\langle G, * \rangle$ *is a group, then left and right cancellation laws hold. That is,* $\forall a, b, c \in G$*,*

$$(a * b = a * c) \Rightarrow b = c \quad \text{and} \quad (b * a = c * a) \Rightarrow b = c.$$

We may use the previous theorem to prove the uniqueness of the following, by assuming there are two different solutions and showing they are the same.

Theorem. *If* $\langle G, * \rangle$ *is a group, then for all* $a, b \in G$*, the equations* $x * a = b$ *and* $a * y = b$ *have unique solutions.*

Theorem. *The identity element and the inverse of each element are unique in a group.*

Corollary. *For all* a, b *in a group* $\langle G, * \rangle$*, we have* $(a * b)' = b' * a'$.

4.3 Finite Groups and Group Tables

Example 4. *What is the smallest finite group?*

Remark. *The identity element is always its own inverse in every group.*

Remark. *When giving a table for a group operation, always list the identity element first.*

Remark. *Since the equations* $a * x = b$ *and* $y * a = b$ *have unique solutions, each element* b *of a group must appear exactly once in each row and each column of the group table.*

Remark. *All groups with one element are isomorphic. All groups with two elements are isomorphic. All groups with three elements are isomorphic. We say "there is only one group of three elements, up to isomorphism."*

Example 5. *A matrix is a rectangular array of numbers. A diagonal matrix is a square matrix whose only nonzero entries lie on the main diagonal, from the upper left to the lower right corner.*

Determine whether the given set of matrices under matrix multiplication is a group.

a) $n \times n$ *diagonal matrices*

b) $n \times n$ *diagonal matrices with diagonal entries* 1 *or* -1.

Example 6. *Recall that if* $\phi : G \to G'$ *is an isomorphism, then* $\phi(e)$ *is the identity element of* G' *when* e *is the identity element of* G*. Prove that if* a *and* a' *are inverse of each other in* G*, then* $\phi(a)$ *and* $\phi(a')$ *are inverse of each other in* G'.