2 Binary Operations

A **binary operation** $*$ on a set S is a function that maps $S \times S$ into S. If $(a, b) \in S$, then $*((a, b)) \in S$, which is equivalent to $a * b$.

Familiar examples of binary operations are addition and multiplication.

If $H \subseteq S$, we say H is closed under $*$ if for all $a, b \in H$, we also have $(a * b) \in H$. Note that S is closed under $*$ by definition of a binary operation.

A binary operation $*$ is said to be **commutative** if $a * b = b * a$ for all $a, b \in S$.

If for all $a, b, c \in S$, $a * (b * c) = (a * b) * c$, we say $*$ is associative on S.

Theorem (Associativity of Composition)**.** *Let S be a set and let f, g, h be functions that map S into S. Then $f \circ (g \circ h) = (f \circ g) \circ h$.*

For a binary operation $*$ on a set S to be valid, we need

1. $*$ to be **well defined**, and
2. S to be closed under $*$.

Example 1. *On $\mathbb{Z}^+$, division $(/)$ is not a binary operation because $\mathbb{Z}^+$ is not closed under division.*

2.1 Tables

We may define a binary operation on a finite set via a table, such that we list the elements of the set across the top and in the same order along the side.

Example 2. *Complete the following table such that $*$ is associative on $S = \{a, b, c, d\}$.*

$*$	a	b	c	d
a	a	b	c	d
b	b	a	c	d
c	c	d	c	d
d				

Example 3. *Determine whether the binary operation $*$ defined on $\mathbb{Q}$ by letting $a * b = ab + 1$ is commutative and whether $*$ is associative.*

3 Isomorphic Binary Structures

A **binary algebraic structure** $\langle S, * \rangle$ is a set S together with a binary operation $*$ on S. Consider binary algebraic structures $\langle S, * \rangle$ and $\langle S', *' \rangle$. We say an **isomorphism** of S with S' is a 1-1 function ϕ mapping S onto S' such that the homomorphism property holds:

$$\forall x, y \in S : \phi(x * y) = \phi(x) *' \phi(y)$$

To show binary structures $\langle S, * \rangle$ and $\langle S', *' \rangle$ are isomorphic:

1. Define the isomorphism function ϕ
2. Show that ϕ is 1-1: $\phi(x) = \phi(y) \Rightarrow x = y$
3. Show that ϕ is onto S': $\forall s' \in S', \exists s \in S$ such that $\phi(s) = s'$
4. Show that ϕ has the homomorphism property: $\forall x, y \in S, \phi(x * y) = \phi(x) *' \phi(y)$

To show two binary structures are not isomorphic we may show that they have different structural properties. A **structural property** of a binary structure is a property that isomorphic structures share. For example, having an identity element is a structural property, whereas name of an element is not.

Suppose $\langle S, *\rangle$ is a binary structure. An element $e \in S$ is called an **identity element for** $*$ if for all $x \in S$, $e * x = x * e = x$.

Theorem (Uniqueness of Identity Element)**.** *A binary structure $\langle S, *\rangle$ has at most one identity element.*

Proof. Suppose e and e' are two identity elements of S. Then

$$e = e * e' = e'$$

where the first equality holds because e' is an identity element and the second equation holds because e is an identity element. □

Theorem. *Suppose $\langle S, *\rangle$ has an identity element e for $*$. If $\phi : S \to S'$ is an isomorphism of $\langle S, *\rangle$ with $\langle S', *'\rangle$, then $\phi(e)$ is an identity element of $\langle S', *'\rangle$.*

Example 4. *Determine whether $\phi(x) = x^2$ is an isomorphism of $\langle \mathbb{Q}, \cdot\rangle$ with $\langle \mathbb{Q}, \cdot\rangle$.*

Example 5. *Determine whether $\phi(r) = 0.5^r$ is an isomorphism of $\langle \mathbb{R}, +\rangle$ with $\langle \mathbb{R}^+, \cdot\rangle$, where $r \in \mathbb{R}$.*

Example 6. *Let F be the set of all functions $f : \mathbb{R} \to \mathbb{R}$ that have derivatives of all orders. Determine whether $\phi(f)(x) = \frac{d}{dx}\left[\int_0^x f(t)\ dt\right]$ is an isomorphism of $\langle F, +\rangle$ with $\langle F, +\rangle$.*

Example 7. *The map $\phi : \mathbb{Z} \to \mathbb{Z}$ defined by $\phi(n) = n + 1$ for $n \in \mathbb{Z}$ is 1-1 and onto $\mathbb{Z}$. Give the definition of a binary operation $*$ on $\mathbb{Z}$ such that ϕ is an isomorphism mapping*

a) $\langle \mathbb{Z}, +\rangle$ onto $\langle \mathbb{Z}, *\rangle$

b) $\langle \mathbb{Z}, *\rangle$ onto $\langle \mathbb{Z}, +\rangle$

In each case, give the identity element for $*$ on $\mathbb{Z}$.

Example 8. *Prove that if $\phi : S \to S'$ is an isomorphism of $\langle S, *\rangle$ with $\langle S', *'\rangle$, then ϕ^{-1} is an isomorphism of $\langle S', *'\rangle$ with $\langle S, *\rangle$.*