

**DO NOT TURN OVER UNTIL
INSTRUCTED TO DO SO.**

NO CALCULATORS PERMITTED.

EXAM TIME IS 170 MINUTES.

THE EXAM CONSISTS OF 10 QUESTIONS.

Your name: _____

Your SID: _____

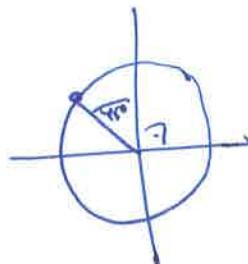
Your Section and GSI: _____

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1. Evaluate the following expressions

- (a) $\sin(135^\circ)$
- (b) $\cos(\frac{5\pi}{3})$
- (c) $\cot(\frac{\pi}{4})$
- (d) $\sin^{-1}(\sin(\frac{7\pi}{8}))$
- (e) $\cos(\tan^{-1}(\frac{1}{3}))$

$$\begin{aligned}
 (a) \quad \sin(135^\circ) &= \sin(90^\circ + 45^\circ) \\
 &= \sin 45^\circ \\
 &= \frac{\sqrt{2}}{2}
 \end{aligned}$$



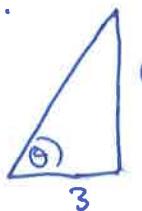
$$\begin{aligned}
 (b) \quad \cos(\frac{5\pi}{3}) &= \cos(\text{fourth quadrant angle}) \cos(\frac{5\pi}{3} - 2\pi) \\
 &= \cos(-\frac{\pi}{3}) \\
 &= \cos(\frac{\pi}{3}) = \frac{1}{2}
 \end{aligned}$$

$$(c) \quad \cot(\frac{\pi}{4}) = \frac{\cos \frac{\pi}{4}}{\sin \frac{\pi}{4}} = 1$$

$$\begin{aligned}
 (d) \quad \sin^{-1}(\sin \frac{7\pi}{8}) &= \frac{7\pi}{8} \quad \text{since} \quad \sin(\frac{7\pi}{8}) = \sin(\frac{\pi}{8}) \\
 &\quad \text{and} \quad \frac{7\pi}{8} \in [-\pi/2, \pi]
 \end{aligned}$$

$$(e) \quad \cos(\tan^{-1}(\frac{1}{3})) = \cos \theta \quad \text{where} \quad \theta = \tan^{-1}(\frac{1}{3}) \quad \text{or} \quad \tan \theta = \frac{1}{3}$$

Draw triangle:



$$\Rightarrow \cos \theta = \frac{3}{\sqrt{3^2 + 1^2}} = \frac{3}{\sqrt{10}}$$

2. (a) Derive the double-angle formula for cosine from the addition theorems.
 (b) Use (a) to derive the half-angle formulae for sine and cosine.
 (c) Compute $\sin(\frac{45^\circ}{2})$.

$$2a) \cos(2\theta) = \cos(\theta + \theta) = \cos^2 \theta - \sin^2 \theta$$

$$= 2\cos^2 \theta - 1 \quad (\text{Pyth.})$$

$$= 1 - 2\sin^2 \theta \quad (\text{Pyth.})$$

$$(b) \cos 2\theta = 2\cos^2 \theta - 1$$

$$\Rightarrow \cos \theta = 2\cos^2 \theta/2 - 1$$

$$\Rightarrow \cos \frac{\theta}{2} = \pm \sqrt{\frac{\cos \theta + 1}{2}}$$

$$\cos 2\theta = 1 - 2\sin^2 \theta$$

$$\Rightarrow \cos \theta = 1 - 2\sin^2 \frac{\theta}{2}$$

$$\Rightarrow \sin \frac{\theta}{2} = \pm \sqrt{\frac{1 - \cos \theta}{2}}$$

$$(c) \sin\left(\frac{45^\circ}{2}\right) = + \sqrt{\frac{1 - \cos 45^\circ}{2}} = \sqrt{\frac{1 - \cos 45^\circ}{2}} = \frac{\sqrt{2 - \sqrt{2}}}{2}$$

↑
 since $\sin \frac{45^\circ}{2} > 0$, see unit circle!

3. Let A, B, C be the angles of a triangle and a, b, c the lengths of the opposite sides respectively.

- (a) Write down the law of sines and explain how to derive it.
- (b) Assuming $a = 5$, $A = \frac{\pi}{6}$ and $B = \frac{\pi}{8}$ find the other angles and sidelengths of the triangle. (Note: You may use your result from Question 2)

$$(a) \text{ Law of sines} \quad \frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

Derivation we have by the area formula

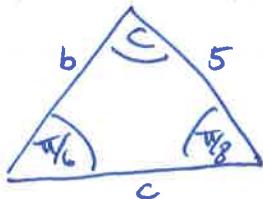
$$A = \frac{1}{2} ab \sin C = \frac{1}{2} bc \sin A = \frac{1}{2} ca \sin B$$

divide every thing by $\frac{1}{2} abc$ to get

$$\frac{\frac{1}{2} ab \sin C}{\frac{1}{2} abc} = \frac{\frac{1}{2} bc \sin A}{\frac{1}{2} abc} = \frac{\frac{1}{2} ca \sin B}{\frac{1}{2} abc}$$

$$\Rightarrow \frac{\sin C}{c} = \frac{\sin A}{a} = \frac{\sin B}{b}$$

(b)



$$\text{we have } C = \pi - \pi/6 - \pi/8 = \frac{17}{24}\pi$$

Now

$$\frac{\sin \pi/6}{5} = \frac{\sin \pi/8}{b} = \frac{\sin \frac{17}{24}\pi}{c}$$

$$\Rightarrow b = 5 \frac{\sin \pi/8}{\sin \pi/6} = 5 \frac{\sqrt{2-\sqrt{2}}}{2 \cdot \frac{1}{2}} = 5 \sqrt{2-\sqrt{2}}$$

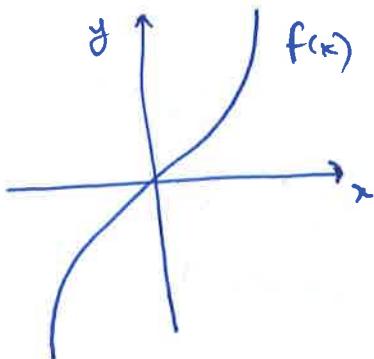
$$\begin{aligned} \Rightarrow c &= 5 \frac{\sin \frac{17}{24}\pi}{\sin \pi/6} = 10 \sin(\pi - \pi/6 - \pi/8) = 10 \sin\left(\frac{5\pi}{6} - \frac{\pi}{8}\right) \\ &= 10 \left(\sin \frac{5\pi}{6} \cos \pi/8 - \sin \pi/8 \cos \frac{5\pi}{6} \right) \\ &= 10 \left(\frac{1}{2} \cos \pi/8 - \frac{\sqrt{2-\sqrt{2}}}{2} \frac{\sqrt{3}}{2} \right) \end{aligned}$$

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see blank page!

4. (a) Let $f(x) = \frac{e^x - e^{-x}}{2}$. Sketch the graph of $f(x)$.
 (b) Use the graph to explain why $f(x)$ is invertible.
 (c) Is $f(x)$ an even or odd function?
 (d) Find the inverse of $f(x)$.

4a)



b) The graph passes horizontal line test \Rightarrow invertible

c) odd function! $f(-x) = \frac{e^{-x} - e^{-(x)}}{2} = \frac{e^{-x} - e^x}{2} = -f(x)$

d) $y = \frac{e^x - e^{-x}}{2} \Rightarrow 2y = e^x - e^{-x}$
 $\Rightarrow 2ye^x = (e^x)^2 - 1$
 $\Rightarrow 2yt = t^2 - 1 \quad \text{where } t = e^x$

Solve for t:

$$t^2 - 2yt - 1 = 0$$

$$\Rightarrow t = \frac{2y \pm \sqrt{4y^2 + 4}}{2} = y \pm \sqrt{y^2 + 1}$$

Now $t = e^x > 0$ so must have plus sign!

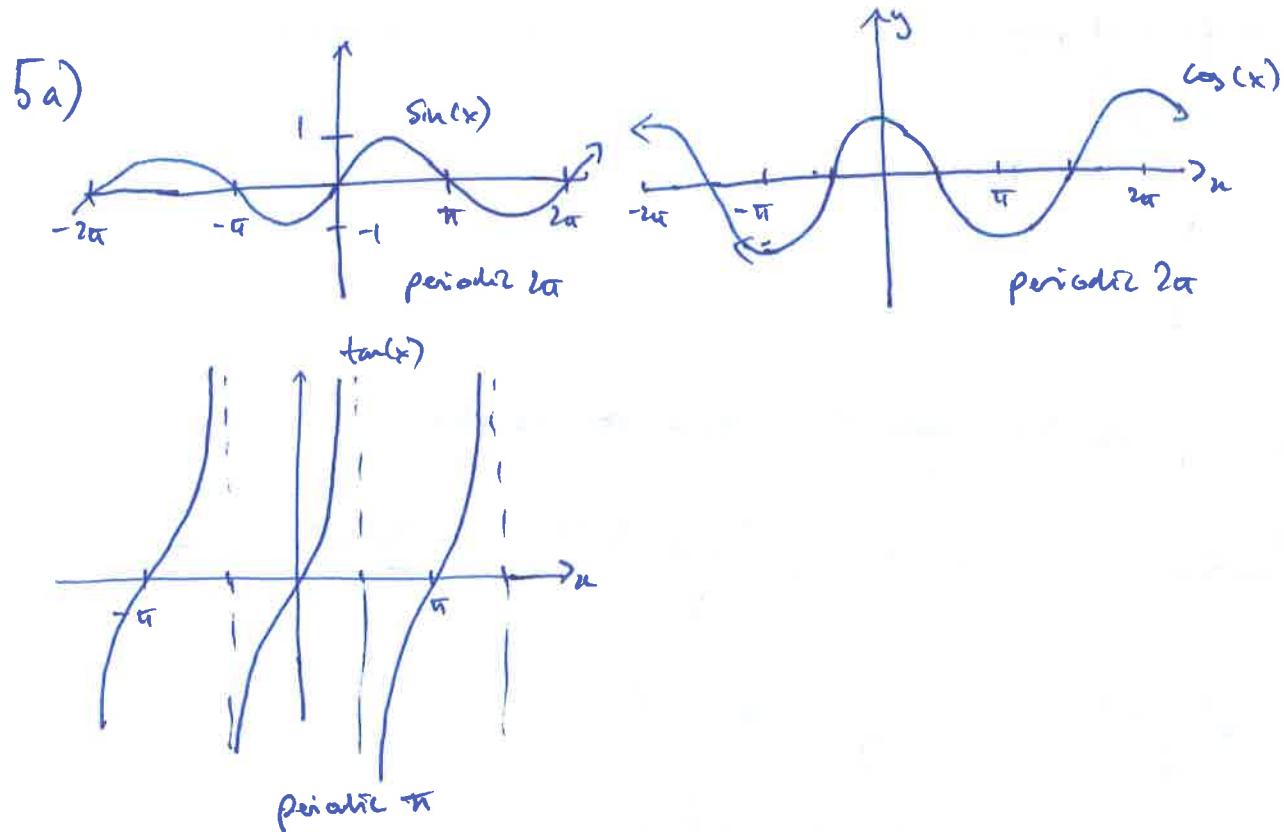
$$\Rightarrow t = y + \sqrt{y^2 + 1} \Rightarrow e^x = y + \sqrt{y^2 + 1}$$

$$\Rightarrow x = \ln(y + \sqrt{y^2 + 1})$$

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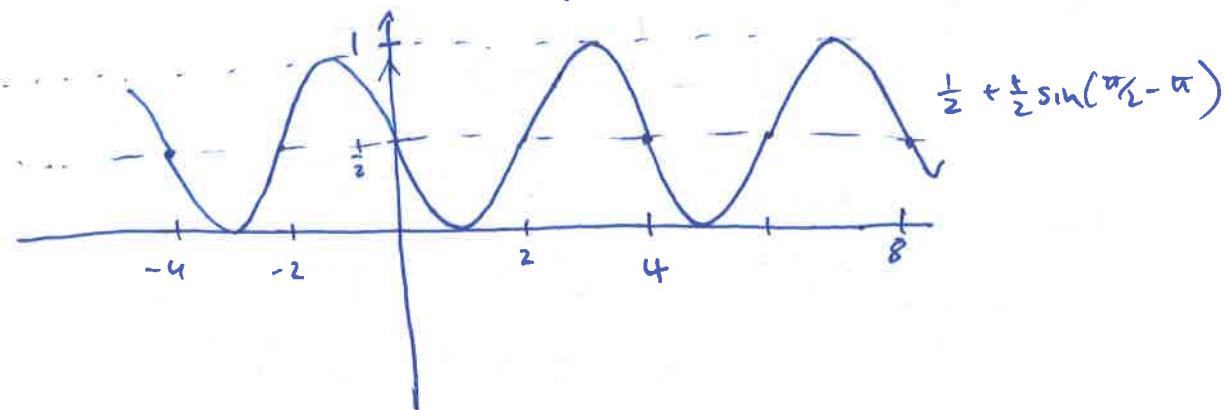
$$\Rightarrow f^{-1}(x) = \ln(x + \sqrt{x^2 + 1})$$

5. (a) Sketch the graphs of $\sin(x)$, $\cos(x)$ and $\tan(x)$.
 (b) How many zeros does $\sin(x)$ have in the interval $(\pi, \frac{27\pi}{2}]$?
 (c) Find the amplitude and the period of the function $f(x) = \frac{1}{2} + \frac{1}{2} \sin(\frac{\pi}{2}x - \pi)$.
 (d) Sketch the function $f(x)$.



b) $\sin x$ has zeros at $n\pi$, $n \in \mathbb{Z} \Rightarrow$ zeros $2\pi, 3\pi, \dots, 13\pi \in (\pi, \frac{27\pi}{2}]$
 $\Rightarrow 12$ zeros in $(\pi, \frac{27\pi}{2}]$

c) amplitude = $\frac{1}{2}$ period = $\frac{2\pi}{\pi/2} = 4$



TURN OVER

(a) Show that $\tan^2(\theta) + 1 = \sec^2(\theta)$

(b) Show using the addition theorem for sine and cosine that

$$\tan(\theta_1 + \theta_2) = \frac{\tan(\theta_1) + \tan(\theta_2)}{1 - \tan(\theta_1)\tan(\theta_2)}$$

(c) Use b) to find an expression for $\tan(\theta_1 - \theta_2)$.

$$(a) \sin^2\theta + \cos^2\theta = 1 \Rightarrow \frac{\sin^2\theta}{\cos^2\theta} + \frac{\cos^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta}$$

$$\Rightarrow \tan^2\theta + 1 = \sec^2\theta$$

$$(b) \tan(\theta_1 + \theta_2) = \frac{\sin(\theta_1 + \theta_2)}{\cos(\theta_1 + \theta_2)} = \frac{\sin\theta_1\cos\theta_2 + \sin\theta_2\cos\theta_1}{\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2}$$

$$= \frac{\cancel{\sin\theta_1\cos\theta_2 + \sin\theta_2\cos\theta_1}}{\cancel{\cos\theta_1\cos\theta_2}} = \frac{\cancel{\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2}}{\cancel{\cos\theta_1\cos\theta_2}}$$

$$= \frac{\frac{\sin\theta_1}{\cos\theta_1} + \frac{\sin\theta_2}{\cos\theta_2}}{1 - \frac{\sin\theta_1}{\cos\theta_1}\frac{\sin\theta_2}{\cos\theta_2}}$$

$$= \frac{\tan\theta_1 + \tan\theta_2}{1 - \tan\theta_1\tan\theta_2}$$

$$(c) \tan(\theta_1 - \theta_2) = \tan(\theta_1 + (-\theta_2)) = \frac{\tan\theta_1 + \tan(-\theta_2)}{1 - \tan\theta_1\tan(-\theta_2)}$$

$$= \frac{\tan\theta_1 - \tan\theta_2}{1 + \tan\theta_1\tan\theta_2}$$

6. (a) Write down the equation for a circle K with centre $C : (2, 3)$ and radius 2.
 (b) Write down the equation of the line l that passes through C and has gradient $-\sqrt{3}$.
 (c) Find the point P of interception of the line l with circle K , for which the y -coordinate is less than 3.
 (d) At what angle does the line l intersect the x -axis?

6a) $(x-2)^2 + (y-3)^2 = 2^2$

b) $\frac{y-3}{x-2} = -\sqrt{3} \Rightarrow y = -\sqrt{3}(x-2) + 3$

c) $(x-2) = \cancel{-\sqrt{3}} - \frac{1}{\sqrt{3}}(y-3) \quad (\text{I})$

and $(x-2)^2 + (y-3)^2 = 2^2 \quad (\text{II})$

Substitute (I) into (II):

$$\left[-\frac{1}{\sqrt{3}}(y-3)\right]^2 + (y-3)^2 = 2^2$$

$$\left(\frac{1}{3}+1\right)(y-3)^2 = 4$$

$$(y-3)^2 = 3$$

$$y = \pm\sqrt{3} + 3$$

$$\Rightarrow y = 3 - \sqrt{3} \quad (\text{as } y < 3)$$

$$\Rightarrow x-2 = -\frac{1}{\sqrt{3}}(3 - \sqrt{3}) = 1 \cancel{\text{---}}$$

$$\Rightarrow x = 3$$

See blank page for d)

7. (a) Derive the base change formula for logarithms
 (b) Compute $\log_{27}(81)$ and $e^{\frac{2}{3}\ln(8)}$.
 (c) Solve $4\log_{16}(x) + 3\log_8(x) + 2\log_2(x) = 4$.

$$7a) \log_a(x) = \log_a(b^{\log_b x}) = \log_b x \log_a b$$

$$\Rightarrow \log_b x = \frac{\log_a x}{\log_a b} \quad (\text{Base change formula})$$

$$b) \log_{27}(81) = \frac{\log_3 81}{\log_3 27} = \frac{4}{3}$$

$$\log e^{\frac{2}{3}\ln(8)} = 8^{\frac{2}{3}} = 2^2 = 4$$

$$c) 4\log_{16}x + 3\log_8(x) + 2\log_2 x = 4$$

$$\Rightarrow 4 \frac{\log_2 x}{\log_2 16} + 3 \frac{\log_2 x}{\log_2 8} + 2 \cancel{\log_2 x} = 4$$

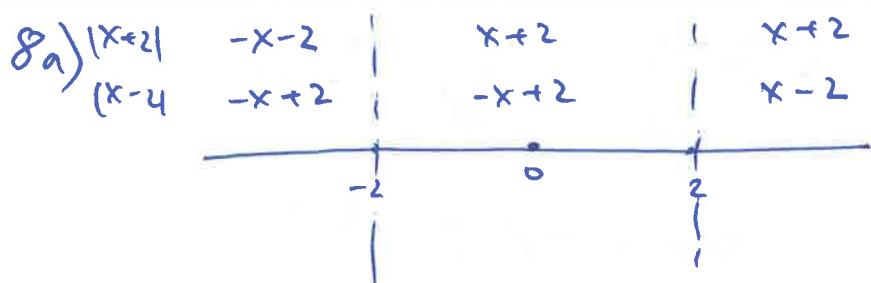
$$\Rightarrow \log_2 x + \log_2 x + 2 \log_2 x = 4$$

$$\Rightarrow 4 \log_2 x = 4$$

$$\Rightarrow \log_2 x = 1$$

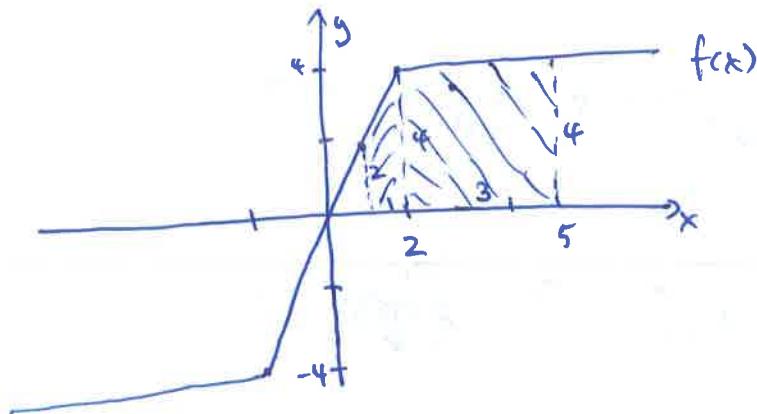
$$\Rightarrow \underline{\underline{x = 2}}.$$

8. (a) Sketch the graph of $f(x) = |x+2| - |x-2|$
 (b) Find the area under the graph of $f(x)$ from $x=1$ to $x=5$
 (c) Find all x for which $2 \leq ||x+2| - |x-2|| \leq 3$



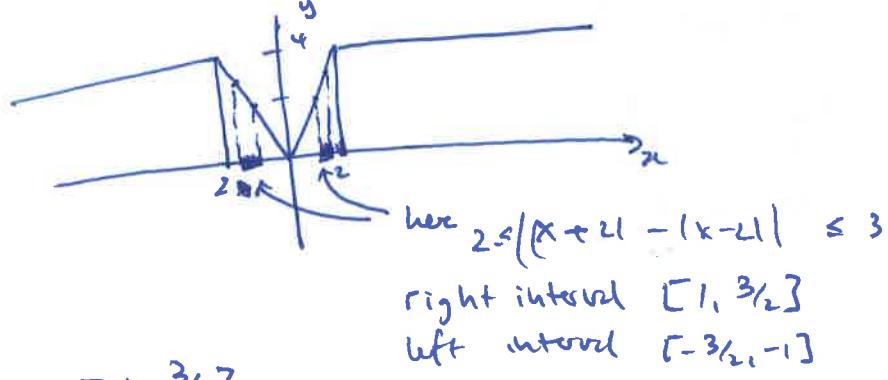
Thus for $x \leq -2$ $f(x) = -x-2 - (-x+2) = -4$
 $-2 \leq x \leq 2$ $f(x) = x+2 - (-x+2) = 2x$
 $x \geq 2$ $f(x) = (x+2) - (x-2) = 4$

Sketch



(b) Area = trapezoid + rectangle $= \frac{2+4}{2} \cdot 1 + 3 \cdot 4 = 15$

(c) $||x+2| - |x-2||:$



⇒ Solution: $[-\frac{3}{2}, -1] \cup [1, \frac{3}{2}]$

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9. (a) Define what a function is.
 (b) Define the terms injective and surjective for functions.
 (c) For the function $f(x) = x^2$ find the pre-image of $(2, 4] \cup [5, 25)$. Write your answer as a union of intervals.

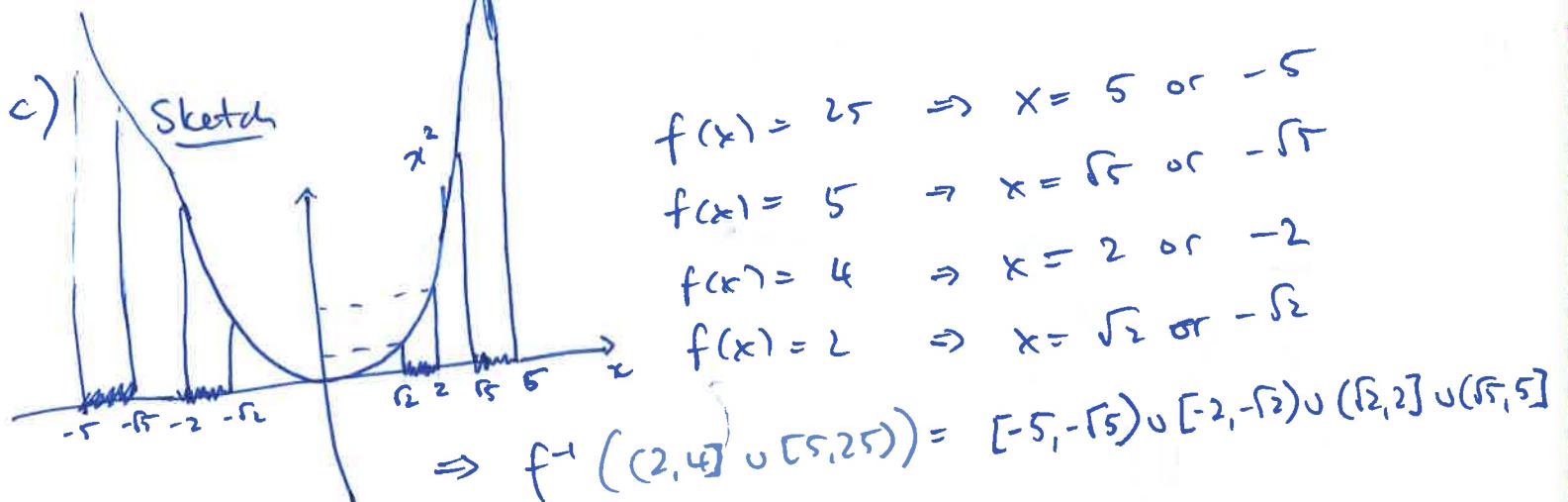
9a) A function consist of three things:

- A domain (set of inputs)
- A codomain (set of outputs)
- A rule, which assigns to each input of the domain a unique output in the codomain

b) injective: Let $f: A \rightarrow B$ be a function

then it's injective, if for all $b \in B$
 there exists at most one $a \in A$ such that
 $f(a) = b$

surjection: $f: A \rightarrow B$ is surjection, if for all $b \in B$
 there exists an $a \in A$ st $f(a) = b$.



10. Let $f(x) = \frac{x^3+x}{x^2+2x+1}$.

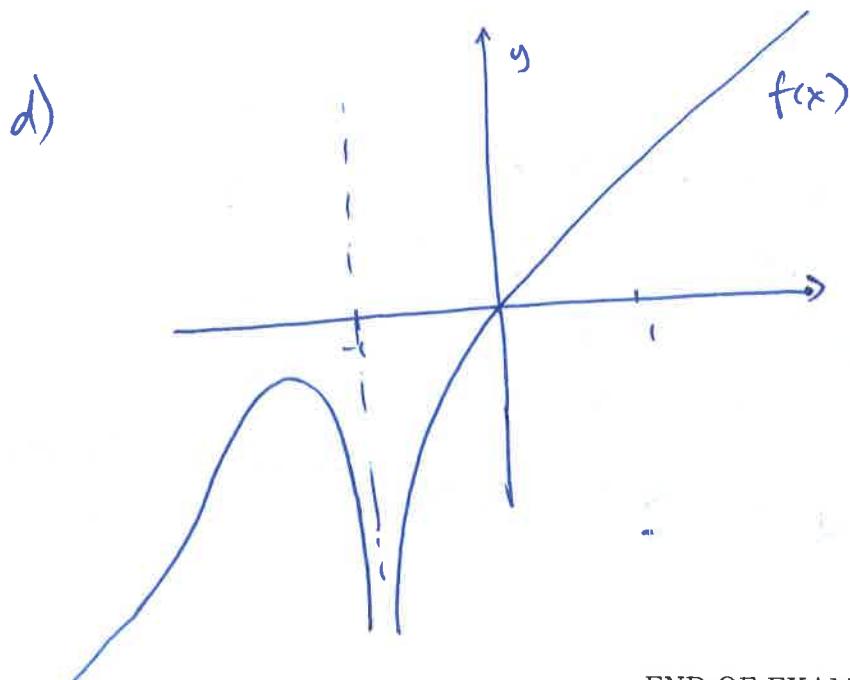
- (a) Find the zeros and vertical asymptotes of $f(x)$. Justify your answer.
- (b) What is the behaviour of $f(x)$ near $\pm\infty$?
- (c) Where is $f(x)$ positive? Express your answer as an interval.
- (d) Sketch the graph of $f(x)$.

| 0a) $f(x)=0 \Leftrightarrow x^3+x=0 \Leftrightarrow x(x^2+1)=0 \Leftrightarrow x=0$
 \uparrow
 only zero

Vertical asymptote: $x^2+2x+1=0$
 $\Leftrightarrow (x+1)^2=0 \Leftrightarrow x=-1$
 \uparrow
 vertical asymptote

b) behavior near $\pm\infty$ = $\frac{\text{leading order term}}{\text{leading order term}} = \frac{x^3}{x^2} = x$

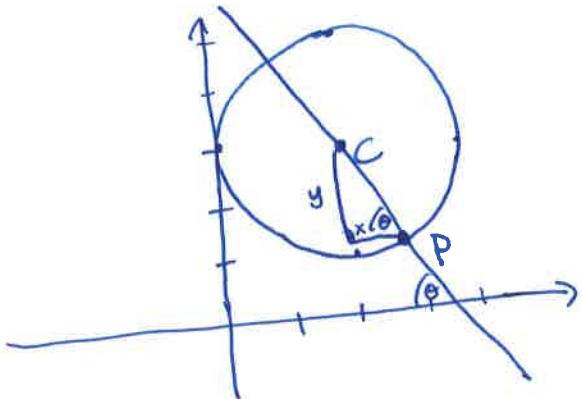
c)
$$\begin{array}{c|ccccc} x & - & & + & & + \\ \hline x^3+x & + & & + & & + \\ (x-1)^2 & + & & + & & + \end{array}$$
 so $f(x)$ is positive on $(0, \infty)$



Continuation of 3b)

$$\begin{aligned} \text{now } \cos \frac{\pi}{8} &= \sqrt{1 - \sin^2 \frac{\pi}{8}} \\ &= \sqrt{1 - \left(\frac{\sqrt{2-\sqrt{2}}}{2}\right)^2} \\ &= \sqrt{1 - \frac{2-\sqrt{2}}{4}} \\ &= \frac{\sqrt{2+\sqrt{2}}}{2} \end{aligned}$$

$$\text{So } c = 10 \left(\frac{1}{2} \frac{\sqrt{2+\sqrt{2}}}{2} - \frac{\sqrt{2-\sqrt{2}}}{2} \frac{\sqrt{3}}{2} \right)$$

Continuation of 6d)Sketch

Draw right triangle as in sketch

$$y = 3 - (3 - \sqrt{3}) = \sqrt{3}$$

$$x = 3 - 2 = 1$$

$$\Rightarrow \tan \theta = \frac{\sqrt{3}}{1} = \sqrt{3}$$

$$\Rightarrow \theta = \tan^{-1}(\sqrt{3}) = 60^\circ$$

Hence the line L intersects the x-axis at an angle of 60° .

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