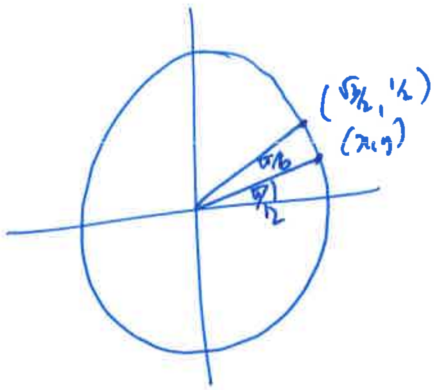
$$(\text{II})' \Rightarrow \quad 2\cos^2\theta/2 = 1 + \cos(\theta)$$

$$\Rightarrow \quad \cos^2\theta/2 = \frac{1+\cos\theta}{2}$$

$$\Rightarrow \quad \cos\theta/2 = \pm\sqrt{\frac{1+\cos\theta}{2}}$$

Again, the sign depends on which quadrant ~~[illegible]~~ $\theta/2$ is in!

<u>Example:</u> Compute $\cos(\pi/12)$ and $\sin(\pi/12)$



$$x = \cos\left(\frac{\pi}{12}\right) = \cos\left(\frac{\pi/6}{2}\right) = \pm\sqrt{\frac{1+\cos\pi/6}{2}}$$

$$\overset{+\text{sign!}}{=} \sqrt{\frac{1+\sqrt{3}/2}{2}} = \sqrt{\frac{\frac{2+\sqrt{3}}{2}}{2}} = \sqrt{\frac{2+\sqrt{3}}{4}}$$

$$y = \sin\left(\frac{\pi}{12}\right) = \sin\left(\frac{\pi/6}{2}\right) = \pm\sqrt{\frac{1-\cos\pi/6}{2}}$$

$$\overset{+\text{sign!}}{=} \sqrt{\frac{1-\frac{\sqrt{3}}{2}}{2}} = \sqrt{\frac{\frac{2-\sqrt{3}}{2}}{2}} = \sqrt{\frac{2-\sqrt{3}}{4}}$$

We have the + sign in both cases because $\pi/12$ is in first quadrant and both sine and cosine are positive here!

This is the nicest known value of sin / cos other than the ones we memorize. Be glad we don't go further!

Angle sum formulas

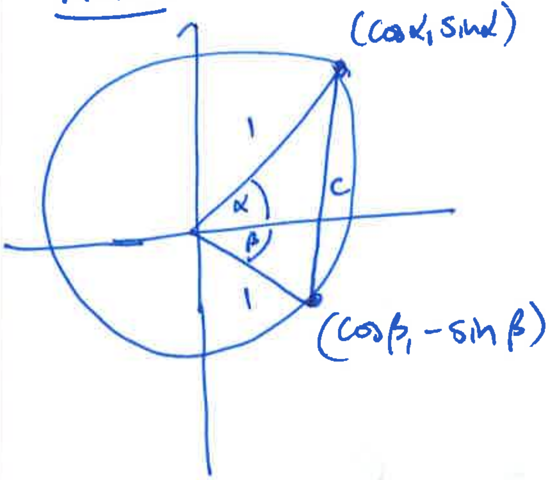
We can find sin / cos (2θ) in terms of $\sin\theta / \cos\theta$.

What about $\sin(\alpha+\beta)$ and $\cos(\alpha+\beta)$?

Ex: $\sin(75^\circ) = \sin(45^\circ + 30^\circ)$, and we know sin / cos (45°) and sin / cos (30°)

We use same trick as for double angle formulae

Picture



Law of cosines:

$$1^2 + 1^2 - 2(1)(1)\cos(\alpha+\beta) = c^2$$

Now c can be computed with the distance formula

$$c = \sqrt{(\cos\alpha - \cos\beta)^2 + (\sin\alpha + \sin\beta)^2}$$

Thus

$$\begin{aligned} c^2 &= (\cos\alpha - \cos\beta)^2 + (\sin\alpha + \sin\beta)^2 \\ &= \cos^2\alpha - 2\cos\alpha\cos\beta + \underline{\cos^2\beta} + \sin^2\alpha + 2\sin\alpha\sin\beta + \underline{\sin^2\beta} \\ &= 1 + 1 - 2\cos\alpha\cos\beta + 2\sin\alpha\sin\beta \end{aligned}$$

Comparing these formulas we get

$$1+1-2\cos\alpha\cos\beta+2\sin\alpha\sin\beta = 1+1-2(1)(1)\cos(\alpha+\beta)$$

Hence $$-2\cos(\alpha+\beta) = -2\cos\alpha\cos\beta + 2\sin\alpha\sin\beta$$

$$\Rightarrow \boxed{\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta}$$ "Addition theorem for cosine"

What about $\sin(\alpha+\beta)$?

Recall: $\sin\theta = \cos(\pi/2-\theta)$ for all θ

So
$$\begin{aligned}
\sin(\alpha+\beta) &= \cos(\pi/2-(\alpha+\beta)) \\
&= \cos((\pi/2-\alpha)-\beta) \\
&= \cos((\pi/2-\alpha)+(-\beta)) \\
&= \cos(\pi/2-\alpha)\cos(-\beta) - \sin(\pi/2-\alpha)\sin(-\beta) \\
&= \sin\alpha\cos(\beta) - \cos\alpha\sin(-\beta)
\end{aligned}$$

$$\Rightarrow \boxed{\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta}$$ (sine odd function, cosine even function

"Addition theorem for sine"

To remember these ~~factors~~ formulas a bit better

- sin is really positive, and inclusive

$$\sin(\alpha)\cos(\beta) + \sin(\beta)\cos(\alpha)$$

- cos is really negative, and divisive, and puts itself first!

$$\cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$$

Once you remember these / Addition theorem / Pythagoras, you get lots of other identities for free

- Angle difference formula

$$\cos(\alpha-\beta) = \cos(\alpha)\cos(-\beta) - \sin(\alpha)\sin(-\beta)$$
$$= \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)$$

$$\sin(\alpha-\beta) = \sin\alpha\cos(-\beta) + \sin(-\beta)\cos(\alpha)$$
$$= \sin\alpha\cos\beta - \sin\beta\cos\alpha$$

- Double angle formulae

$$\cos 2\theta = \cos(\theta+\theta) = \cos\theta\cos\theta - \sin\theta\sin\theta$$
$$= \cos^2\theta - \sin^2\theta$$

$$\sin 2\theta = \sin(\theta+\theta) = \sin\theta\cos\theta + \sin\theta\cos\theta$$
$$= 2\sin\theta\cos\theta$$

- other forms of $\cos(2\theta)$

$$\cos 2\theta = \cos^2\theta - \sin^2\theta = (1-\sin^2\theta) - \sin^2\theta = 1 - 2\sin^2\theta$$

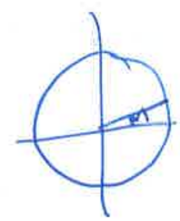
$$\cos 2\theta = \cos^2\theta - \sin^2\theta = \cos^2\theta - (1-\cos^2\theta) = 2\cos^2\theta - 1$$

- half-angle formulas

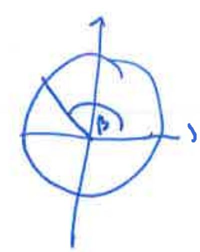
$$\cos\theta = 1 - 2\sin^2(\theta/2), \quad \text{solve for } \sin(\theta/2)$$

$$\cos\theta = 2\cos^2\theta/2 - 1, \quad \text{solve for } \cos(\theta/2)$$

<u>Example:</u> $\sin(\alpha) = \frac{1}{4}$ and $\alpha \in [0, \pi/2] \rightarrow$



$\sin(\beta) = \frac{2}{3}$ $\quad \beta \in [\pi/2, \pi] \rightarrow$



Find $\sin(\alpha+\beta)$

Addition theorem: $\sin(\alpha+\beta) = \underset{\frac{1}{4}}{\underset{\updownarrow}{\sin\alpha}}\, \underset{?}{\cos\beta} + \underset{\frac{2}{3}}{\underset{\updownarrow}{\sin\beta}}\, \underset{?}{\cos\alpha}$

<u>Pythagoras:</u>

$$\sin^2\alpha + \cos^2\alpha = 1$$

$$\Rightarrow \cos\alpha = \pm\sqrt{1-\sin^2\alpha}$$

$$\Rightarrow \cos\alpha = \sqrt{1-\sin^2\alpha} \quad \text{(see picture why + sign)}$$

$$\Rightarrow \cos\alpha = \sqrt{1-\left(\tfrac{1}{4}\right)^2} = \frac{\sqrt{15}}{4}$$

Similarly $\sin^2\beta + \cos^2\beta = 1$

$$\Rightarrow \cos\beta = \pm\sqrt{1-\sin^2\beta}$$

$$= \pm\sqrt{1-\left(\frac{2}{3}\right)^2}$$

$$= -\sqrt{1-\frac{4}{9}}$$ (− sign, ~~because of~~ see ~~picture~~ picture!)

$$= -\frac{\sqrt{5}}{3}$$

hence $\sin(\alpha+\beta) = \left(\frac{1}{4}\right)\left(-\frac{\sqrt{5}}{3}\right) + \left(\frac{\sqrt{15}}{4}\right)\left(2/3\right) = -\frac{\sqrt{5}}{12} + \frac{2\sqrt{15}}{12} = \frac{2\sqrt{15}-\sqrt{5}}{12}$

if time, show $\cos(75^\circ) = \cos(45^\circ + 30^\circ) \ldots = \frac{\sqrt{6}-\sqrt{2}}{4}$

$$\cos(75^\circ) = \cos(90^\circ - 15^\circ) = \sin 15^\circ = \sin\frac{30^\circ}{2}$$

$$\left(\sin\theta/2 = \pm\sqrt{\frac{1-\cos\theta}{2}}\right)$$