Math 1B : Calculus (Fall 2014)

Homework 9 Solutions

§ 9.4

Q3 a) This is the logistic model so a general solution is given by $y(t) = \frac{M}{1 + Ae^{-kt}}$ for $A$ any constant.

If we know $P_0 = y(0)$ then $A = \frac{M - P_0}{P_0}$

In this case $M = 8 \times 10^7$, $P_0 = 2 \times 10^7$ $\Rightarrow$

$A = \frac{6 \times 10^7}{2 \times 10^7} = 3$. Also $k = 0.71$. Hence

$$y(t) = \frac{8 \times 10^7}{1 + 3e^{-0.71t}}$$

Hence $y(1) = \frac{8 \times 10^7}{1 + 3e^{-0.71}} \approx 3.23 \times 10^7$ kg.

b) Need to solve

$$4 \times 10^7 = \frac{8 \times 10^7}{1 + 3e^{-0.71t}}$$

$$\Rightarrow \quad 2 = \frac{1}{1 + 3e^{-0.71t}} \Rightarrow t \approx 1.55 \text{ years}$$

Q4 a) Must solve $kP\left(1-\frac{P}{M}\right) = 0.4P - 0.001P^2$

$$\Rightarrow \quad kP - \frac{kP^2}{M} = 0.4P - 0.001P^2$$

$$\Rightarrow \quad k = 0.4 \quad \text{and} \quad \frac{k}{M} = 0.001$$

$$\Rightarrow \quad M = \frac{0.4}{0.001} = 400$$

b) $P'(0) = (0.4)\cdot 50 - 0.001\cdot(50)^2$

$$= 20 - 2.5 = 17.5$$

c) solution is

$$P(t) = \frac{M}{1+Ae^{-kt}} = \frac{400}{1+7e^{-0.4t}}$$

$$\left(A = \frac{M-P_0}{P_0}\right)$$

Need to solve $200 = P(t) = \frac{400}{1+7e^{-0.4t}}$

$$\Rightarrow \quad 1+7e^{-0.4t} = 2 \quad \Rightarrow \quad e^{-0.4t} = \frac{1}{7}$$

$$\Rightarrow \quad t = \frac{-\ln\left(\frac{1}{7}\right)}{0.4} \approx 4.86 \text{ years.}$$

Q11 a) $\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right)$

$$\Rightarrow \frac{d^2P}{dt^2} = \frac{d\left(kP\left(1-\frac{P}{M}\right)\right)}{dt} \overset{\text{Chain rule}}{=} \frac{d\left(kP\left(1-\frac{P}{M}\right)\right)}{dP} \cdot \frac{dP}{dt}$$

$$= \left(k - \frac{2k}{M}P\right)\left(kP\left(1-\frac{P}{M}\right)\right)$$

$$= k^2P\left(1-\frac{P}{M}\right)\left(1-\frac{2P}{M}\right).$$

b) $\frac{dP}{dt}$ reaches it maximum value when

$$kP\left(1-\frac{P}{M}\right)$$

reaches a maximum. But $kP - \frac{k}{M}P^2$ is a quadratic with negative coefficient in front of the $P^2$ term. Hence the max value occurs when

$$k - \frac{2k}{M}P = \frac{d\left(kP\left(1-\frac{P}{M}\right)\right)}{dP} = 0 \quad \Rightarrow \quad P = \frac{M}{2}.$$

Hence maximal growth is at $P = \frac{M}{2}$. //

Q15

a) $\frac{dP}{dt} = kP - m$. Separable, hence must solve

$$\int \frac{dP}{kP-m} = \int dt$$

$$\Rightarrow \quad \frac{1}{k}\ln|kP-m| = t + C$$

$$\Rightarrow \quad |kP-m| = e^{kt+C'} \qquad (C' = kC)$$

$$\Rightarrow \quad kP = \pm e^{C'} e^{kt} + m$$

$$\Rightarrow \quad P = Ae^{kt} + \frac{m}{k} \qquad (A \text{ any constant.})$$

If $P(0) = P_0 \Rightarrow \quad P_0 = A + \frac{M}{k} \Rightarrow \quad A = \frac{kP_0 - m}{k}$

$$\Rightarrow \quad P(t) = \frac{kP_0 - m}{k} e^{kt} + \frac{m}{k}$$ //

b) $\frac{kP_0 - m}{k} > 0 \iff kP_0 - m > 0 \iff P_0 > \frac{m}{k}$.

c) $\frac{kP_0 - m}{k} < 0 \iff P_0 < \frac{m}{k}$ $\in$ decline

$\frac{kP_0 - m}{k} = 0 \iff P_0 = \frac{m}{k}$ $\in$ constant.

d) Decline.

Q21 / a) Separable, hence must solve

$$\int \frac{dP}{P} = \int k \cos(rt-\phi)\, dt$$

$$\Rightarrow \quad \ln|P| = \frac{k}{r}\sin(rt-\phi) + C$$

$$\Rightarrow \quad P = A e^{\frac{k}{r}\sin(rt-\phi)} \quad (A \text{ any constant})$$

$$(A = P_0 e^{-\frac{k}{r}\sin(-\phi)} = P_0 e^{\frac{k}{r}\sin(\phi)})$$

b) Larger $k$ gives bigger oscillation in population size.

Larger $r$ will decrease scale of population oscillation as well as increase oscillation frequency.

$\phi$ just translates the graph.

The limit does <u>not</u> exist.

§ 9.5

Q5 $y' + y = 1$

Choose $\int p(x)\,dx = \int dx = x \Rightarrow I(x) = e^x$

$$\Rightarrow \frac{d(e^x y)}{dx} = e^x \Rightarrow e^x y = e^x + C$$

$$\Rightarrow y = 1 + \frac{C}{e^x} = 1 + Ce^{-x}$$ //

Q11 Note $\sin(x)\frac{dy}{dx} + \cos(x)\,y = \frac{d(\sin(x)\,y)}{dx}$

$$\Rightarrow \frac{d(\sin(x)\,y)}{dx} = \sin(x^2)$$

$$\Rightarrow y = \frac{1}{\sin(x)} \int \sin(x^2)\,dx$$ //

Q20 $(x^2+1)\frac{dy}{dx} + 3x(y-1) = 0$

$$\Rightarrow \frac{dy}{dx} + \frac{3x}{(x^2+1)}\,y = \frac{3x}{x^2+1}$$

$\int \frac{3x}{x^2+1}\,dx = \frac{3}{2}\ln(x^2+1) + C$. Choose $I(x) = e^{\frac{3}{2}\ln(x^2+1)}$

$= (x^2+1)^{\frac{3}{2}}$

$$\Rightarrow \frac{d\left((x^2+1)^{\frac{3}{2}} y\right)}{dx} = 3x(x^2+1)^{\frac{1}{2}}$$

$$\Rightarrow (x^2+1)^{\frac{3}{2}} y = \int 3x(x^2+1)^{\frac{1}{2}}\,dx = (x^2+1)^{\frac{3}{2}} + C$$

$$\Rightarrow \quad y = 1 + \frac{c}{(x^2+1)^{\frac{3}{2}}}$$

$$y(0) = 2 \quad \Rightarrow \quad 2 = 1 + \frac{c}{1} \quad \Rightarrow \quad c = 1$$

$$\Rightarrow \quad y(x) = 1 + \frac{1}{(x^2+1)^{\frac{3}{2}}} \quad //$$

Q23/ $u = y^{1-n} \qquad (n > 1)$

$$\Rightarrow \quad y = u^{\frac{1}{(1-n)}}$$ (Chain Rule)

$$\Rightarrow \quad \frac{dy}{dx} = \frac{d\, u^{\frac{1}{1-n}}}{du} \cdot \frac{du}{dx}$$

$$= \frac{1}{(1-n)} u^{\left(\frac{1}{1-n} - 1\right)} \cdot \frac{du}{dx} = \frac{1}{1-n} u^{\left(\frac{n}{1-n}\right)} \cdot \frac{du}{dx}$$

$$\frac{dy}{dx} + P(x)\, y = Q(x)\, y^n$$

$$\Rightarrow \quad \frac{1}{1-n} u^{\left(\frac{n}{1-n}\right)} \cdot \frac{du}{dx} + P(x)\, u^{\frac{1}{1-n}} = Q(x)\, u^{\frac{n}{1-n}}$$

$$\Rightarrow \quad \frac{1}{1-n} \frac{du}{dx} + P(x)\, u^{\left(\frac{1}{1-n} - \frac{n}{1-n}\right)} = Q(x)$$

$$\Rightarrow \quad \frac{du}{dx} + (1-n) P(x)\, u = (1-n) Q(x) \quad //$$

Q25 Let $u = y^{1-3} = y^{-2}$ $\left(y = u^{-\frac{1}{2}}\right)$

Need to solve

$$\frac{du}{dx} + (1-3)\frac{2}{x} \cdot u = \frac{(1-3)}{x^2}$$

$$\Rightarrow \quad \frac{du}{dx} - \frac{4}{x} u = \frac{-2}{x^2}$$

$$\int \frac{-4}{x} dx = -4 \ln|x| + C.$$

Choose $I(x) = e^{-4\ln(x)} = e^{\ln(x^{-4})} = x^{-4}$

$$\Rightarrow \quad \frac{d\, x^{-4} u}{dx} = -2x^{-6}$$

$$\Rightarrow \quad x^{-4} u = \int -2x^{-6} dx = \frac{2}{5} x^{-5} + C$$

$$\Rightarrow \quad u = \frac{2}{5x} + Cx^4$$

$$\Rightarrow \quad y = \frac{1}{\sqrt{\frac{2}{5x} + Cx^4}}$$

Q27 a) Need to solve

$$2\frac{dI}{dt} + 10I = 40$$

$$\Rightarrow \frac{dI}{dt} + 5I = 20$$

Multiply both sides by $e^{\int 5\,dt} = e^{5t}$, gives

$$\frac{d(e^{5t}I)}{dt} = 20e^{5t}$$

$$\Rightarrow e^{5t}I(t) = 4e^{5t} + C$$

$$\Rightarrow I(t) = 4 + \frac{C}{e^{5t}}$$

$I(0) = 0 \Rightarrow 0 = 4 + \frac{C}{e^{5\cdot 0}} \Rightarrow 0 = 4 + C \Rightarrow C = -4$

$$\Rightarrow I(t) = 4 - \frac{4}{e^{5t}}$$

b) $I(0.1) = 4 - \frac{4}{e^{0.5}} \approx 1.57\ A.$

Q29 $5\frac{dQ}{dt} + \frac{1}{0.05}Q = 60$

$$\Rightarrow \frac{dQ}{dt} + 4Q = 12$$

Multiply both sides by $e^{\int 4\,dt} = e^{4t}$

Giving

$$\frac{d(e^{4t}Q(t))}{dt} = 12e^{4t}$$

$$\Rightarrow \quad Q(t) = e^{-4t}(3e^{4t} + C)$$

$$= 3 + \frac{C}{e^{4t}}$$

$Q(0) = 0 \quad \Rightarrow \quad C = -3$

$$\Rightarrow \quad Q(t) = 3 - \frac{3}{e^{4t}} \quad //$$

$$\frac{dQ(t)}{dt} = 12e^{-4t} = I(t) \quad //$$

§ 9.6

Q1 / a) $x$ = predators

$y$ = prey

If $x = 0$ $\frac{dy}{dt} = 0.1y$ so growth is not restricted in absence of predators.

If $y = 0$ $\frac{dx}{dt} = -0.05x$, so population declines. This suggest the prey are their only source of food.

b) $x$ = prey

$y$ = predators

If $x = 0$ $\frac{dy}{dt} = 0.015y$. So population declines. This suggest $x$ is only source of food for $y$.

If $y = 0$ $\frac{dx}{dt} = 0.2x - 0.0002x^2$

This is logistic model so $y$ grows exponentially for small values but is limited by the environment in some way independent of $x$.

Q2 / a) $x$ and $y$ cooperate because $\frac{dx}{dt}$ increases if $y$ increases and $\frac{dy}{dt}$ increases if $x$ increases.

b) $x$ and $y$ compete because $\frac{dx}{dt}$ decreases if $y$ increases and $\frac{dx}{dt}$ decreases if $y$ increases.

Q3 a) Competition

b) Need $x$ and $y$ such that $\frac{dx}{dt} = \frac{dy}{dt} = 0$

$$\Rightarrow \quad 0.5x - 0.004x^2 - 0.001xy = 0 \quad \text{and}$$

$$0.4y - 0.001y^2 - 0.001xy = 0$$

If $y = 0 \Rightarrow 0.5x - 0.004x^2 = 0 \Rightarrow x = 0$

or

$$x = \frac{0.5}{0.004}$$

$$= 125$$

If $x = 0 \Rightarrow 0.4y - 0.001y^2 = 0 \Rightarrow y = 0$

or

$$y = \frac{0.4}{0.001}$$

$$= 400$$

If $x \neq 0$ and $y \neq 0 \Rightarrow$

$$0.5 - 0.004x - 0.001y = 0 \quad \text{and}$$

$$0.4 - 0.001y - 0.0.001x = 0$$

Solving then gives $x = 50$, $y = 300$.

$x = 0$, $y = 0$ zero population

$x = 0$, $y = 400$ $y$ is at carrying capacity

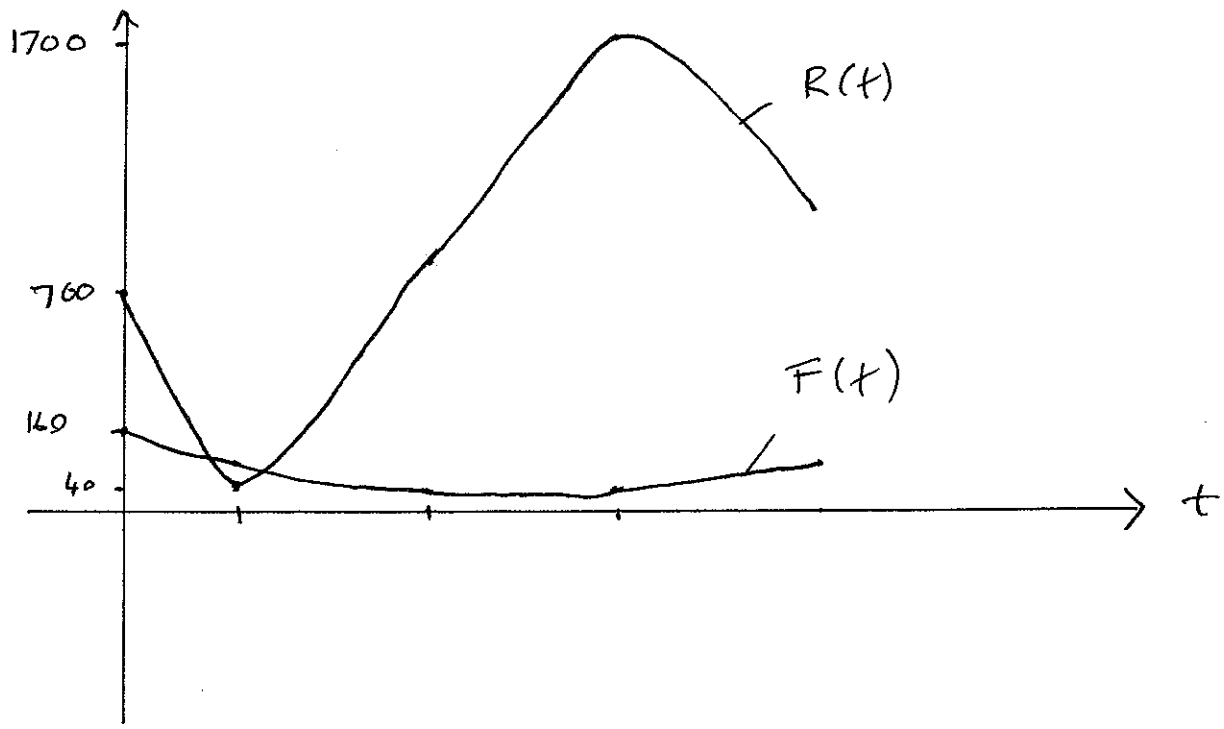
$x = 125$, $y = 0$ $x$ is at carrying capacity

$x = 50$, $y = 300$ Both non-zero and stable.

Q5/ See answers in back of the textbook.

Q6/a) At $t=0$, the fox population is 160 and the rabbit population is roughly 700. From here the rabbit population rapidly drops to about 50. In this time the fox population drops to about 80. It continues to fall to just under 40, when the rabbit population explodes going up to over 1600. Only then does the fox population start to increase going up to 70 by which time the rabbit population has gone down to roughly 900.

b)



Q10

a) $2A - 0.01AL = 0$

$-0.5L + 0.0001AL = 0$

$\Rightarrow$ Either $A = 0, L = 0$ or $A = \dfrac{0.5}{0.0001} = 5000$

↗ zero populations

$L = \dfrac{2}{0.01} = 200$

↑ Stable populations.

b) $\dfrac{dL}{dA} = \dfrac{\frac{dL}{dt}}{\frac{dA}{dt}} = \dfrac{-0.5L + 0.0001AL}{2A - 0.01AL}$

c) See back of book for phase portraits.

All phase trajectories are closed loops.