



Q51 Recall that  $\lim_{x \rightarrow \infty} \arctan(x) = \frac{\pi}{2}$ . Also

$$\lim_{n \rightarrow \infty} \{\ln(n)\} = \infty, \text{ hence } \lim_{n \rightarrow \infty} \{\arctan(\ln(n))\} = \frac{\pi}{2} //$$

Q69 Let  $r > 0$  and consider the function  $f(x) = x r^x$   
( $r \neq 1$ )  
 $= x e^{(\ln(r)x)}$

$$\lim_{x \rightarrow \infty} x e^{(\ln(r)x)} = \lim_{x \rightarrow \infty} \frac{x}{e^{-\ln(r)x}} = \lim_{x \rightarrow \infty} \frac{1}{-\ln(r) e^{-\ln(r)x}}$$

$$= \lim_{x \rightarrow \infty} \frac{r^x}{-\ln(r)} = \begin{cases} \infty & \text{if } r > 1 \\ 0 & \text{if } r < 1. \end{cases}$$

For  $r=1$  then  $n r^n = n$ , which is divergent.

Hence sequence convergent if  $0 < r < 1$  and divergent for  $r \geq 1$ . If  $r < 0$ , observe  $|n r^n| = n |r|^n$ . We see that if  $-1 < r \leq 0$  the sequence converges to 0. If

$r \leq -1$  the sequence diverges.

Q70 a) Assume  $\lim_{n \rightarrow \infty} \{a_n\} = a$ . Then given  $\varepsilon > 0$  there

exists an integer  $N$  such that  $|a_n - a| < \varepsilon$  for  $n > N$ .

Hence  $|a_{n+1} - a| < \varepsilon$  for  $n+1 > N$ , that is  $n > N-1$ .

This implies that  $\lim_{n \rightarrow \infty} \{a_{n+1}\} = a$ . //

b) Assume  $\lim_{n \rightarrow \infty} \{a_n\} = a$ . By part (a), because

$$a_{n+1} = \frac{1}{1+a_n}, \text{ we know } a = \frac{1}{1+a} \Rightarrow$$

$$a = \frac{-1 \pm \sqrt{5}}{2}$$

Note that  $a_n > 0$  for all  $n$ , implying that  $a > 0$ .

$$\text{Hence } a = \frac{-1 + \sqrt{5}}{2}$$

Q71 // By Monotone Sequence Theorem we know  $\{a_n\}$  is convergent. Because  $a_n \geq 5$  for all  $n$ ,  $\lim_{n \rightarrow \infty} \{a_n\} \geq 5$ .

Because  $\{a_n\}$  decreases and  $a_1 \leq 8$ , we also know that

$$\lim_{n \rightarrow \infty} \{a_n\} < 8.$$

Q79 //

$$a_1 = \sqrt{2}$$

$$a_2 = \sqrt{2\sqrt{2}} = \sqrt{2a_1}$$

$$a_3 = \sqrt{2\sqrt{2\sqrt{2}}} = \sqrt{2a_2}$$

$$\Rightarrow a_{n+1} = \sqrt{2a_n}$$

By Q70 (c) if  $a = \lim_{n \rightarrow \infty} \{a_n\}$  then

$$a = \sqrt{2a} \Rightarrow a^2 = 2a$$

$$\Rightarrow a = 0 \text{ or } 2.$$

We must work out which it is. Note that if  $2 > a_n > 0$

$\Rightarrow$

$$2a_n > a_n^2 \Rightarrow \sqrt{2a_n} = a_{n+1} > a_n$$

Also note that  $2 > a_n \Rightarrow 4 > 2a_n \Rightarrow 2 > \sqrt{2a_n} = a_{n+1}$ .

$2 > \sqrt{2} = a_1$ . Hence by induction  $\{a_n\}$  is bounded above by 2 and is increasing. Hence it is convergent to 2. //