

Math 1B: Calculus (Fall 2014)

Homework 11 Solutions

§17.2

Q3
 $y'' + 9y = e^{-2x}$

First solve $y'' + 9y = 0$

$$r^2 + 9 = 0 \Rightarrow \begin{matrix} \alpha = 0 \\ \beta = 3 \end{matrix} \quad \left(\begin{matrix} r_1 = 3i \\ r_2 = -3i \end{matrix} \right)$$

$$\Rightarrow y_c(x) = c_1 \cos(3x) + c_2 \sin(3x).$$

Try $y_p(x) = A e^{-2x}$

$$y_p'' + 9y_p = (4A + 9A)e^{-2x} = 13A e^{-2x}$$

Need $13A = 1 \Rightarrow A = \frac{1}{13}$.

General solution is

$$y(x) = \frac{1}{13} e^{-2x} + c_1 \cos(3x) + c_2 \sin(3x) //$$

Q7, First solve $y'' + y = 0$

$$r^2 + 1 = 0 \Rightarrow \begin{matrix} \alpha = 0 \\ \beta = 1 \end{matrix} \quad \left(\begin{matrix} r_1 = i \\ r_2 = -i \end{matrix} \right)$$

$$\Rightarrow y_c(x) = c_1 \cos(x) + c_2 \sin(x)$$

Solve $y'' + y = e^x$

Try $y_{p_1}(x) = A e^x \Rightarrow y_{p_1}''(x) + y_{p_1}(x) = 2A e^x$

Let $A = \frac{1}{2}$.

Solve $y'' + y = x^3$.

Try $y_{p2}(x) = Ax^3 + Bx^2 + Cx + D$

$$y_{p2}''(x) + y_{p2}(x) = Ax^3 + Bx^2 + (C + 6A)x + (D + 2B)$$

$$\begin{array}{l} \text{Need } A = 1 \\ B = 0 \\ C + 6A = 0 \\ D + 2B = 0 \end{array} \Rightarrow \begin{array}{l} A = 1 \\ B = 0 \\ C = -6 \\ D = 0 \end{array}$$

$$\Rightarrow y_{p2}(x) = x^3 - 6x$$

General solution is $y(x) = \frac{1}{2}e^x + x^3 - 6x + c_1 \cos(x) + c_2 \sin(x)$

$$y(0) = 1 \Rightarrow \frac{1}{2} + c_1 = 1 \Rightarrow c_1 = \frac{1}{2}$$

$$y'(x) = \frac{1}{2}e^x + 3x^2 - 6 - c_1 \sin(x) + c_2 \cos(x)$$

$$y'(0) = \frac{1}{2} - 6 + c_2 = 0 \Rightarrow c_2 = 6 - \frac{1}{2} = \frac{11}{2}$$

Thus $y(x) = \frac{1}{2}e^x + x^3 - 6x + \frac{1}{2} \cos(x) + \frac{11}{2} \sin(x)$

$$8/ \text{Solve } y'' - 4y = e^x \cos(x).$$

$$\text{First solve } y'' - 4y = 0$$

$$r^2 - 4 = 0 \Rightarrow \begin{aligned} r_1 &= 2 \\ r_2 &= -2 \end{aligned}$$

$$\Rightarrow \text{general solution is } y_c(x) = c_1 e^{2x} + c_2 e^{-2x}$$

For particular solution try

$$y_p(x) = e^{2x} (A \cos(x) + B \sin(x))$$

$$y_p'' - 4y_p = e^x \left((2B - 4A) \cos(x) + (-2A - 4B) \sin(x) \right)$$

$$\begin{aligned} \text{Need } 2B - 4A &= 1 \\ -2A - 4B &= 0 \end{aligned} \Rightarrow \begin{aligned} A &= \frac{-2}{10} \\ B &= \frac{1}{10} \end{aligned}$$

General solution is

$$y(x) = e^x \left(\frac{-2}{10} \cos(x) + \frac{1}{10} \sin(x) \right)$$

$$+ c_1 e^{2x} + c_2 e^{-2x}$$

$$y(0) = 1$$

$$\Rightarrow 1 = \frac{-2}{10} + c_1 + c_2$$

$$y'(x) = e^x \left(\frac{-1}{10} \cos(x) + \frac{3}{10} \sin(x) \right) + 2c_1 e^{2x} - 2c_2 e^{-2x}$$

$$y'(0) = 2 \Rightarrow$$

$$2 = \frac{-1}{10} + 2c_1 - 2c_2$$

Thus

$$c_1 + c_2 = \frac{17}{10} \Rightarrow c_1 = \left(\frac{12}{10} + \frac{21}{20} \right) = \frac{45}{40}$$
$$c_1 - c_2 = \frac{21}{20}$$

$$c_2 = \frac{12}{10} - \frac{45}{40} = \frac{3}{40}$$

Hence

$$y(x) = e^x \left(\frac{-2}{10} \cos(x) + \frac{1}{10} \sin(x) \right)$$

$$+ \frac{45}{40} e^{2x} + \frac{3}{40} e^{-2x}$$

Q9/ First solve $y'' - y' = 0$

$$r^2 - r = 0 \Rightarrow r_1 = 1$$
$$r_2 = 0$$

$$y_c(x) = c_1 e^x + c_2$$

The trial solution $y_p(x) = (Ax + B)e^x$ has a term (Be^x) which is a solution to the complementary equation. Hence multiply trial solution by x .

$$y_p(x) = (Ax^2 + Bx)e^x$$

$$y_p'(x) = (Ax^2 + (2A+B)x + B)e^x$$

$$\begin{aligned} y_p''(x) &= (Ax^2 + (2A+2A+B)x + (2A+B+B))e^x \\ &= (Ax^2 + (4A+B)x + (2A+2B))e^x \end{aligned}$$

$$y_p'' - y_p' = (2Ax + (2A+B))e^x$$

$$\begin{aligned} \text{Need } 2A &= 1 & \Rightarrow A &= \frac{1}{2} \\ 2A+B &= 0 & \Rightarrow B &= -1 \end{aligned}$$

Thus a general solution is $y(x) = \left(\frac{1}{2}x^2 - x\right)e^x + c_1 e^x + c_2$

$$y(0) = 2 \Rightarrow$$

$$2 = c_1 + c_2$$

$$y'(x) = \left(\frac{1}{2}x^2 + 0 \cdot x - 1\right)e^x + c_1 e^x$$

$$y'(0) = 1 \Rightarrow$$

$$-1 + c_1 = 1 \Rightarrow c_1 = 2$$

$$\Rightarrow c_2 = 0$$

$$\text{Hence } y(x) = \left(\frac{1}{2}x^2 - x\right)e^x + 2e^x //$$

Q15/ e^x is a solution to the complementary equation

hence we should take as a trial solution

$$y_p(x) = Ax^2 e^x + B \cos(x) + C \sin(x) //$$

Q17/ $y'' + 2y' + 10y = 0$ has general solution

$$y_c(x) = C_1 e^{-x} \cos(3x) + C_2 e^{-x} \sin(3x)$$

If we choose trial solution

$$y_p(x) = (Ax^2 + Bx + C) e^{-x} (D \cos(3x) + E \sin(3x))$$

then when we expand the function some terms will be solutions to complementary equation. Hence we must

instead try $y_p(x) = x(Ax^2 + Bx + C) e^{-x} (D \cos(3x) + E \sin(3x))$

None of the terms in this expansion are ~~in~~ complementary solutions. Hence we are done //

§17.4

Q1/ Let $y(x) = \sum_{n=0}^{\infty} c_n x^n \Rightarrow y'(x) = \sum_{n=0}^{\infty} n c_n x^{n-1}$

Rewrite this as $y'(x) = \sum_{n=0}^{\infty} (n+1) c_{n+1} x^n$

$y' - y = 0 \Rightarrow \sum_{n=0}^{\infty} ((n+1) c_{n+1} - c_n) x^n = 0 \Rightarrow$

$(n+1) c_{n+1} = c_n$ for all $n \geq 0$.

But this means $c_n = \frac{c_0}{n!} \Rightarrow$

$y(x) = c_0 \sum_{n=0}^{\infty} \frac{x^n}{n!} = c_0 e^x //$

Q5/ Let $y(x) = \sum_{n=0}^{\infty} c_n x^n \Rightarrow$

$y'(x) = \sum_{n=0}^{\infty} (n+1) c_{n+1} x^n$

$y''(x) = \sum_{n=0}^{\infty} (n+2)(n+1) c_{n+2} x^n$

$\Rightarrow x y'(x) = \sum_{n=0}^{\infty} (n+1) c_{n+1} x^{n+1} = \sum_{n=1}^{\infty} n c_n x^n$

$y'' + x y' + y = 0 \Rightarrow \sum_{n=0}^{\infty} (n+2)(n+1) c_{n+2} x^n + \sum_{n=1}^{\infty} n c_n x^n + \sum_{n=0}^{\infty} c_n x^n = 0$

Thus we get the final solution

$$y(x) = c_0 \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n n!} x^{2n} + c_1 \sum_{n=0}^{\infty} \frac{(-2)^n n!}{(2n+1)!} x^{2n+1}$$

Q9 / Using the same notation as Q5 we get

$$y'' - xy' - y = (2c_2 - c_0) + \sum_{n=1}^{\infty} ((n+2)(n+1)c_{n+2} - nc_n - c_n) x^n$$

$$y'' - xy' - y = 0$$

$$\Rightarrow 2c_2 - c_0 = 0 \text{ and for all } n \geq 1$$

$$(n+2)c_{n+2} - c_n = 0$$

$$\text{Hence } c_{n+2} = \frac{c_n}{n+2} \text{ for all } n.$$

As in Q5 we deduce the general solution

$$y(x) = c_0 \sum_{n=0}^{\infty} \frac{x^{2n}}{2^n n!} + c_1 \sum_{n=0}^{\infty} \frac{2^n n!}{(2n+1)!} x^{2n+1}$$

$$y(0) = 1 \Rightarrow c_0 = 1$$

$$y'(0) = 0 \Rightarrow c_1 = 0 \text{ (The only } x \text{ term is } c_1 x)$$

$$\text{Hence } y(x) = \sum_{n=0}^{\infty} \frac{x^{2n}}{2^n n!} = \sum_{n=0}^{\infty} \frac{\left(\frac{x^2}{2}\right)^n}{n!} = e^{\frac{x^2}{2}}$$

Q11 / Using the same notation as Q5/ we get

$$y''(x) = \sum_{n=0}^{\infty} (n+2)(n+1) c_{n+2} x^n$$

$$x^2 y'(x) = \sum_{n=2}^{\infty} (n-1) c_{n-1} x^n$$

$$xy(x) = \sum_{n=1}^{\infty} c_{n-1} x^n$$

$$\Rightarrow y'' + x^2 y' + xy = 2c_2 + (2 \cdot 3 \cdot c_3 + c_0) x + \sum_{n=2}^{\infty} \left((n+2)(n+1) c_{n+2} + n c_{n-1} \right) x^n = 0$$

$$\Rightarrow 2c_2 = 0 \Rightarrow c_2 = 0.$$

Note that for all $n \geq 0$ we have

$$(n+3)(n+2)c_{n+3} + (n+1)c_n = 0$$

$$c_2 = 0 \Rightarrow c_{3n+2} = 0 \text{ for all } n \geq 0.$$

$$c_4 = \frac{-2}{3 \cdot 4} c_1, \quad c_7 = \frac{(-5) \cdot (-2)}{(3 \cdot 6) \cdot (4 \cdot 7)} c_1 \\ = \frac{(-1)^2 2^2 \cdot 5^2}{7!} c_1$$

More generally (for $n \geq 1$)

$$c_{3n+1} = \frac{(-1)^n 2^2 \cdot 5^2 \cdots (3n-1)^2}{(3n+1)!} c_1$$

Similarly

$$c_3 = \frac{-1}{3 \cdot 2} c_0, \quad c_6 = \frac{(-1) \cdot (-4)}{(3 \cdot 6) \cdot (2 \cdot 5)} c_0$$
$$= \frac{(-1)^2 \cdot 1^2 \cdot 4^2}{6!} c_0$$

($n \geq 1$)

More generally

$$c_{3n} = \frac{(-1)^n 1^2 \cdot 4^2 \dots (3n-2)^2}{(3n)!} c_0$$

Hence a general solution is

$$y(x) = c_0 \left(1 + \sum_{n=1}^{\infty} \frac{(-1)^n 1^2 \cdot 4^2 \dots (3n-2)^2}{(3n)!} x^{3n} \right)$$
$$+ c_1 \left(x + \sum_{n=1}^{\infty} \frac{(-1)^n 2^2 \cdot 5^2 \dots (3n-1)^2}{(3n+1)!} x^{3n+1} \right)$$

$$y(0) = 0 \Rightarrow c_0 = 0$$

$$y'(0) = 1 \Rightarrow c_1 = 1$$

Hence $y(x) = x + \sum_{n=1}^{\infty} \frac{(-1)^n 2^2 \cdot 5^2 \dots (3n-1)^2}{(3n+1)!} x^{3n+1}$

Q12(a) /

$$x^2 y'' = \sum_{n=2}^{\infty} n(n-1) C_n x^n$$

$$x y' = \sum_{n=1}^{\infty} n C_n x^n$$

$$x^2 y = \sum_{n=2}^{\infty} C_{n-2} x^n$$

$$\Rightarrow x^2 y'' + x y' + x^2 y = C_1 x + \sum_{n=2}^{\infty} (n^2 C_n + C_{n-2}) x^n = 0$$

$$\Rightarrow C_1 = 0 \quad \text{Because } n^2 C_n + C_{n-2} = 0 \text{ for all } n \geq 2$$

we deduce that $C_n = 0$ for all n odd.

$$C_2 = \frac{-1}{2^2} C_0, \quad C_4 = \frac{(-1)^2}{2^2 \cdot 4^2} C_0 \quad \text{More generally}$$

$$C_{2n} = \frac{(-1)^n}{2^2 \cdot 4^2 \cdots 2n^2} C_0 = \frac{(-1)^n}{(2^n n!)^2} C_0 = \frac{(-1)^n}{2^{2n} (n!)^2} C_0$$

This gives the solution

$$y(x) = C_0 \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n} (n!)^2}$$

This is a little strange because we expect to get another linearly independent solution. What is going on is that there is another linearly independent solution (called a Bessel Function of the 2nd kind), but it does not have a

power series representation around 0. This is why this method is only picking up the usual Bessel function power series. Thankfully the conditions

$y(0) = 1$, $y'(0) = 0$ are satisfied by

$$y(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n} (n!)^2}$$