Math 1B : Calculus.

Homework 1 Solutions.

Section 7.1

Q5 $f(t) = t$, $g'(t) = e^{-3t}$. Choose $g(t) = \frac{-1}{3}e^{-3t}$. $\Rightarrow$

$$\int te^{-3t}\,dt = \frac{-1}{3}te^{-3t} - \int \frac{-1}{3}e^{-3t}\,dt = \frac{-1}{3}te^{-3t} + \frac{-1}{9}e^{-3t} + C$$

Q7 $f(x) = x^2+2x$, $g'(x) = \cos(x)$. Choose $g(x) = \sin(x)$ $\Rightarrow$

$$\int (x^2+2x)\cos x\,dx = (x^2+2x)\sin(x) - \int (2x+2)\sin(x)\,dx.$$

Repeat with $f(x) = 2x+2$ $g'(x) = \sin(x)$. Choose $g(x) = -\cos(x)$ $\Rightarrow$

$$\int (x^2+2x)\cos x\,dx = (x^2+2x)\sin(x) - \left((2x+2)(-\cos(x)) - \int 2(-\cos(x))\,dx\right)$$

$$= (x^2+2x)\sin(x) + (2x+2)\cos(x) - 2\sin(x) + C$$

Q11 $f(t) = \arctan(4t)$, $g'(t) = 1$, Choose $g(t) = t$ $\Rightarrow$

$$\int \arctan(4t)\,dt = t\arctan(4t) - \int \frac{4t}{(4t)^2+1}\,dt$$

Substitute $16t^2 = u$ $\Rightarrow$ $\frac{du}{dt} = 32t$ $\Rightarrow$ $dt = \frac{1}{32t}\,du$ $\Rightarrow$

$$\int \frac{4t}{(4t)^2+1}\,dt = \frac{1}{8}\int \frac{1}{u+1}\,du = \frac{1}{8}\ln|16t^2+1| + C \Rightarrow$$

$$\int \arctan(4t)\,dt = t\arctan(4t) - \frac{1}{8}\ln|16t^2+1| + C$$

Q15 $f(x) = (\ln(x))^2$, $g'(x) = 1$. Choose $g(x) = x$.

$$\int (\ln(x))^2\,dx = x\ln(x)^2 - \int x\,\frac{2\ln(x)}{x}\,dx = x\ln(x)^2 - 2\int \ln(x)\,dx$$

$$= x\ln(x)^2 - 2x\ln(x) + 2x + C$$

Q17 $f(\theta) = e^{2\theta}$, $g'(\theta) = \sin(3\theta)$. Choose $g(\theta) = \frac{-1}{3}\cos(3\theta)$ $\Rightarrow$

$$\int e^{2\theta}\sin 3\theta\,d\theta = e^{2\theta}\left(\frac{-1}{3}\cos(3\theta)\right) - \int 2e^{2\theta}\cdot\left(\frac{-1}{3}\cos(3\theta)\right)d\theta$$

$$= -\frac{e^{2\theta}}{3}\cos(3\theta) + \frac{2}{3}\int e^{2\theta}\cos(3\theta)\,d\theta$$

Repeat with $f(\theta) = e^{2\theta}$, $g'(\theta) = \cos(3\theta)$. Choose $g(\theta) = \frac{1}{3}\sin(3\theta)$

$$\Rightarrow \int e^{2\theta}\sin(3\theta)\,d\theta = -\frac{e^{2\theta}}{3}\cos(3\theta) + \frac{2}{3}\left(\frac{e^{2\theta}}{3}\sin(3\theta) - \int \frac{2e^{2\theta}}{3}\sin(3\theta)\,d\theta\right)$$

$$= -\frac{e^{2\theta}}{3}\cos(3\theta) + \frac{2}{9}e^{2\theta}\sin(3\theta) - \frac{4}{9}\int e^{2\theta}\sin(3\theta)\,d\theta$$

$$\Rightarrow \int e^{2\theta}\sin(3\theta)\,d\theta = \frac{9}{13}\left(\frac{2}{9}e^{2\theta}\sin(3\theta) - \frac{e^{2\theta}}{3}\cos(3\theta)\right) + C$$

Q40 Observe that $\sin 2t = 2\sin t\cos t$. Hence substitute $u = \cos t$

$$\Rightarrow \frac{du}{dt} = -\sin(t) \Rightarrow dt = \frac{1}{-\sin(t)}\,du$$

$$\int_0^{\pi} e^{\cos(t)}\sin(2t)\,dt = -\int_1^{-1} 2u e^u\,du = \int_{-1}^{1} 2ue^u\,du.$$

Use IBP. $f(u) = 2u$, $g'(u) = e^u$. Choose $g(u) = e^u \Rightarrow$

$$\int_{-1}^{1} 2ue^u\,du = \Big[2ue^u\Big]_{-1}^{1} - \int_{-1}^{1} 2e^u\,du = 2e + 2e^{-1} - [2e^u]_{-1}^{1}$$
$$= 4e^{-1}$$

Q41 Let $u = 1 + x \Rightarrow \frac{du}{dx} = 1 \Rightarrow du = dx \Rightarrow$

$$\int x\ln(1+x)\,dx = \int (u-1)\ln u\,du = \int u\ln u\,du - \int \ln u\,du$$

We do both of these by integration by parts

$$\int \underset{g'(u),\ g(u) = \frac{1}{2}u^2}{u}\,\underset{f(u)}{\ln(u)}\,du = \frac{1}{2}u^2\ln(u) - \int \frac{1}{2}u\,du = \frac{1}{2}u^2\ln(u) - \frac{1}{4}u^2 + C$$

$$\Rightarrow \int x\ln(1+x)\,dx = \frac{1}{2}(x+1)^2\ln(x+1) - \frac{1}{4}(1+x)^2 - (x+1)\ln(x+1) + (x+1) + C$$

Q48 a) Do integration by parts with $f(x) = \cos^{n-1}(x)$, $g'(x) = \cos(x)$.
Choose $g(x) = \sin(x)$

$$\Rightarrow \int \cos^n(x)\,dx = \cos^{n-1}(x)\sin(x) - \int (-\sin^2(x))(n-1)\cos^{n-2}(x)\,dx$$

$$= \cos^{n-1}(x)\sin(x) + (n-1)\int (1-\cos^2(x))\cos^{n-2}(x)\,dx$$

$$= \cos^{n-1}(x)\sin(x) + (n-1)\int \cos^{n-2}(x)\,dx - (n-1)\int \cos^n(x)\,dx$$

$$\Rightarrow \int \cos^n(x)\,dx = \frac{1}{n}\cos^{n-1}(x)\sin(x) + \frac{n-1}{n}\int \cos^{n-2}\,dx$$

b) By part a) (with $n=2$) we have

$$\int \cos^2(x)\,dx = \frac{1}{2}\cos(x)\sin(x) + \frac{1}{2}\int 1\cdot dx = \frac{1}{2}\cos(x)\sin(x) + \frac{1}{2}x + C$$

c) By part a) (with $n=4$) we have

$$\int \cos^4(x)\,dx = \frac{1}{4}\cos^3(x)\sin(x) + \frac{3}{4}\left(\frac{1}{2}\cos(x)\sin(x) + \frac{1}{2}x\right) + C$$

Q 50 By the reduction formula on page 468 we know that

$$\int_0^{\frac{\pi}{2}} \sin^{2n}x\,dx = \left[-\frac{1}{2n}\cos(x)\sin^{2n-1}(x)\right]_0^{\frac{\pi}{2}} + \frac{2n-1}{2n}\int_0^{\frac{\pi}{2}} \sin^{2n-2}(x)\,dx$$

$$= 0 + \frac{2n-1}{2n}\int_0^{\frac{\pi}{2}} \sin^{2n-2}(x)\,dx$$

Assume $\int_0^{\frac{\pi}{2}} \sin^{2n-2}(x)\,dx = \frac{1\cdot 3\cdot 5\cdots 2n-3}{2\cdot 4\cdot 6\cdots 2n-2}\cdot\frac{\pi}{2}$

$$\Rightarrow \int_0^{\frac{\pi}{2}} \sin^{2n}x\,dx = \frac{2n-1}{2n}\cdot\left(\frac{1\cdot 3\cdot 5\cdots 2n-3}{2\cdot 4\cdot 6\cdots 2n-2}\cdot\frac{\pi}{2}\right)$$

$$= \frac{1\cdot 3\cdot 5\cdots 2n-1}{2\cdot 4\cdots 2n}\cdot\frac{\pi}{2}$$

Note that $\int_0^{\frac{\pi}{2}} \sin^0(x)\,dx = \int_0^{\frac{\pi}{2}} 1\,dx = \frac{\pi}{2}$.

By induction on n, we get the result.

Q51 Let $f(x) = \ln(x)^n$, $g'(x) = 1$. Choose $g(x) = x$ $\Rightarrow$

$$\int \ln(x)^n dx = x \ln(x)^n - \int x \cdot n \cdot \frac{1}{x} \ln(x)^{n-1} dx$$

$$= x \ln(x)^n - n \int \ln(x)^{n-1} dx$$

Section 7.2

Q2 $\int \sin^3\theta \cos^4\theta \, d\theta$. Power of sin is odd so substitute

$u = \cos\theta \Rightarrow \frac{du}{d\theta} = -\sin\theta \Rightarrow d\theta = \frac{1}{-\sin\theta}\, du$

$= \int \sin^3\theta \cos^4\theta \, d\theta = \int \sin^2\theta \, u^4 \, du = \int (1-u^2) u^4 \, du$

$= \frac{1}{5}u^5 - \frac{1}{7}u^7 + C = \frac{1}{5}\cos^5(x) - \frac{1}{7}\cos^7(x) + C.$

Q6 Try substitution $u = \sqrt{x} \Rightarrow \frac{du}{dx} = \frac{1}{2}x^{-\frac{1}{2}} \Rightarrow$

$dx = 2\sqrt{x}\, du \Rightarrow$

$\int \frac{\sin^3(\sqrt{x})}{\sqrt{x}}\, dx = 2\int \sin^3(u)\, du$. Substitute $v = \cos(u)$

$\Rightarrow \frac{dv}{du} = -\sin(u) \Rightarrow du = \frac{1}{-\sin(u)}\, dv \Rightarrow$

$2\int \sin^3(u)\, du = -2\int \sin^2(u)\, dv = -2\int (1-v^2)\, dv$

$= -2v + \frac{2}{3}v^3 + C = -2\cos(u) + \frac{2}{3}\cos^3(u) + C$

$= -2\cos(\sqrt{x}) + \frac{2}{3}\cos^3(\sqrt{x}) + C$

Q7 $\int_0^{\frac{\pi}{2}} \cos^2\theta \, d\theta = \int_0^{\frac{\pi}{2}} \frac{1+\cos(2\theta)}{2}\, d\theta = \frac{1}{2}\left[\theta + \frac{1}{2}\sin(2\theta)\right]_0^{\frac{\pi}{2}}$

$= \frac{\pi}{4}$

Q10 $\int_0^{\pi} \sin^2 t \cos^4 t \, dt = \int_0^{\pi} (1-\cos^2 t)\cos^4 t = \int_0^{\pi} \cos^4(t)\, dt - \int_0^{\pi} \cos^6(t)\, dt$

$\int_0^{\pi} \cos^4(t)\, dt = \int_0^{\pi} \left(\frac{1+\cos(2t)}{2}\right)^2 dt$

$= \frac{1}{4}\int_0^{\pi} 1 + 2\cos(2t) + \left(\frac{1+\cos(4t)}{2}\right) dt$

$$= \frac{1}{4}\left[\frac{3}{2}t + \sin(2t) + \frac{1}{8}\sin(4t)\right]_0^{\pi}$$

$$= \frac{3\pi}{8}$$

$$\int_0^{\pi} \cos^6(t)\,dt = \int_0^{\pi} \left(\frac{1+\cos(2t)}{2}\right)^3 dt$$

$$= \frac{1}{8}\int_0^{\pi} \left(1 + 3\cos(2t) + 3\cos^2(2t) + \cos^3(2t)\right) dt$$

$$= \frac{1}{8}\int_0^{\pi} 1 + 3\cos(2t) + \frac{3}{2}\left(1+\cos(4t)\right) dt$$

$$+ \frac{1}{8}\int_0^{\pi} \cos^3(2t)\,dt \quad \leftarrow \text{substitute } u = \sin(2t)$$

$$= \frac{1}{8}\left[\frac{5}{2}t + \frac{3}{2}\sin(2t) + \frac{3}{8}\sin(4t)\right]_0^{\pi}$$

$$+ \frac{1}{8}\int_0^{0} \frac{1}{2}(1-u^2)\,du$$

$$= \frac{5\pi}{16}$$

$$\Rightarrow \int_0^{\pi} \sin^2(t)\cos^4(t)\,dt = \frac{3\pi}{8} - \frac{5\pi}{16} = \frac{\pi}{16} \quad //$$

Q14 //

$$\int \frac{\cos x + \sin 2x}{\sin(x)}\,dx = \int \cot(x) + \frac{2\sin(x)\cos(x)}{\sin(x)}\,dx$$

$$= \int \cot(x)\,dx + 2\int \cos(x)\,dx$$

$$= \ln|\sin(x)| + 2\sin(x) + C \quad //$$

Q24

$$\int (\tan^2 x + \tan^4(x))\,dx = \int \tan^2(x)\,(1+\tan^2(x))\,dx$$

$$= \int \sec^2(x)\tan^2(x)\,dx.$$

Substitute $u = \tan(x) \Rightarrow \frac{du}{dx} = \sec^2(x) \Rightarrow dx = \frac{1}{\sec^2(x)}\,du$

$$\Rightarrow \int (\tan^2(x) + \tan^4(x))\,dx = \int u^2\,du = \frac{1}{3}u^3 + C$$

$$= \frac{1}{3}\tan^3(x) + C$$

Q31

$$\int \tan^5(x)\,dx = \int \sin^5(x)\cos^{-5}(x)\,dx.$$ Substitute

$u = \cos(x) \Rightarrow \frac{du}{dx} = -\sin(x) \Rightarrow dx = \frac{1}{-\sin(x)}\,du \Rightarrow$

$$\int \tan^5(x)\,dx = -\int \sin^4(x)\,u^{-5}\,du = -\int (1-u^2)^2 u^{-5}\,du$$

$$= -\int (u^{-5} - 2u^{-3} + u^{-1})\,du = \frac{-1}{-4}u^{-4} - \frac{2}{2}u^{-2} - \ln|u| + C$$

$$= \frac{1}{4}\cos^{-4}(x) - \cos^{-2}(x) - \ln|\cos(x)| + C$$

Q33

$$\int x\sec(x)\tan(x)\,dx$$

$f(x)$ $\quad$ $g'(x)$, choose $g(x) = \sec(x)$

$$x\sec(x) - \int \sec(x)\,dx = x\sec(x) - \ln|\sec(x)+\tan(x)| + C$$

Q43

$$\int \sin(5\theta)\sin\theta\,d\theta = \frac{1}{2}\int (\cos 4\theta - \cos 6\theta)\,d\theta$$

$$= \frac{1}{8}\sin 4\theta - \frac{1}{12}\sin 6\theta + C$$

Q48

$$\int \frac{1}{\cos(x)-1}\,dx = \int \frac{\cos(x)+1}{(\cos(x)-1)(\cos(x)+1)}\,dx$$

$$= \int \frac{\cos(x)+1}{-\sin^2(x)}\,dx$$

$$= -\int \sin^{-2}(x)\cos(x)\,dx - \int \csc^2(x)\,dx$$

$$= \frac{-1}{-1}\sin^{-1}(x) - (-\cot(x)) + C$$

$$= \csc(x) + \cot(x) + C$$

Q68

$$\int_{-\pi}^{\pi} \cos(mx)\cos(nx)\,dx = \frac{1}{2}\int_{-\pi}^{\pi} \cos((m-n)x) + \cos((m+n)x)\,dx$$

If $m-n \neq 0$ (ie $m \neq n$) this equals

$$\frac{1}{2}\left[\frac{1}{(m-n)}\sin((m-n)(x)) + \frac{1}{m+n}\sin((m+n)(x))\right]_{-\pi}^{\pi} = \frac{1}{2}\left((0+0)-(0+0)\right)$$

$$= 0$$

If $m=n$ this equals

$$\frac{1}{2}\int_{-\pi}^{\pi} 1 + \cos(2mx)\,dx = \frac{1}{2}\left[x + \frac{1}{2m}\sin(2m(x))\right]_{-\pi}^{\pi}$$

$$= \frac{1}{2}\left((\pi+0)-(-\pi+0)\right)$$

$$= \frac{2\pi}{2} = \pi$$