DO NOT TURN OVER UNTIL INSTRUCTED TO DO SO.

Formulae

$$\int \tan(x)\,dx = \ln|\sec(x)| + C \qquad \int \sec(x)\,dx = \ln|\sec(x)+\tan(x)| + C$$

$$\int \frac{1}{1+x^2}dx = \arctan(x) + C \qquad \int \frac{1}{\sqrt{1-x^2}}dx = \arcsin(x) + C$$

$$\frac{d\tan(x)}{dx} = \sec^2(x) \qquad \frac{d\sec(x)}{dx} = \tan(x)\sec(x)$$

$$1 = \sin^2(x) + \cos^2(x) \qquad 1 + \tan^2(x) = \sec^2(x)$$

$$\cos^2(x) = \frac{1+\cos(2x)}{2} \qquad \sin^2(x) = \frac{1-\cos(2x)}{2}$$

$$|E_M| \leq \frac{K(b-a)^3}{24n^2} \qquad |E_S| \leq \frac{K(b-a)^5}{180n^4}$$

CALCULATORS ARE NOT PERMITTED

YOU MAY USE YOUR OWN BLANK PAPER FOR ROUGH WORK

SO AS NOT TO DISTURB OTHER STUDENTS, EVERYONE MUST STAY UNTIL THE EXAM IS FINISHED

REMEMBER THIS EXAM IS GRADED BY A HUMAN BEING. WRITE YOUR SOLUTIONS NEATLY AND COHERENTLY, OR THEY RISK NOT RECEIVING FULL CREDIT

Math 1B First Midterm Exam (LEC 001) 8:10 am to 9am

This exam consists of 5 questions. Answer the questions in the spaces provided.

Name and section: ______________________

GSI's name: ______________________

1. Compute the following integrals:

   (a) (10 points)

$$\int x \arctan(x)\, dx$$

Solution:

Do integration by parts with $f(x) = \arctan(x)$, $g'(x) = x$, $g(x) = \frac{1}{2}x^2$. Hence

$$\int x\arctan(x)\,dx = \frac{1}{2}x^2\arctan(x) - \frac{1}{2}\int \frac{x^2}{x^2+1}\,dx$$

$$= \frac{1}{2}x^2\arctan(x) - \frac{1}{2}\left(\int 1\,dx - \int \frac{1}{x^2+1}\,dx\right)$$

$$= \frac{1}{2}x^2\arctan(x) - \frac{1}{2}x + \frac{1}{2}\arctan(x) + C$$

   (b) (10 points)

$$\int x^3\sqrt{1-x^2}\, dx$$

Solution:

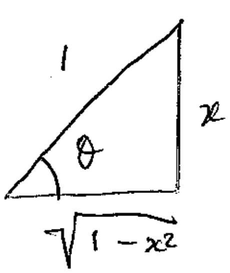
Substitute $x = \sin\theta$ $\left(-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}\right)$ $\Rightarrow \frac{dx}{d\theta} = \cos\theta$

$\Rightarrow dx = \cos\theta\, d\theta \Rightarrow \int x^3\sqrt{1-x^2}\,dx = \int \sin^3\theta\cos^2\theta\,d\theta$

Substitute $u = \cos\theta \Rightarrow \frac{du}{d\theta} = -\sin\theta \Rightarrow d\theta = \frac{1}{-\sin\theta}\,du$

$\Rightarrow \int \sin^3\theta\cos^2\theta\,d\theta = -\int(1-u^2)u^2\,du = \frac{-1}{3}u^3 + \frac{1}{5}u^5 + C$

$= \frac{-1}{3}\cos^3\theta + \frac{1}{5}\cos^5\theta + C = -\frac{1}{3}(1-x^2)^{\frac{3}{2}} + \frac{1}{5}(1-x^2)^{\frac{5}{2}} + C$



PLEASE TURN OVER

$\Rightarrow \cos\theta = \sqrt{1-x^2}$

3. (a) (10 points) Express the following rational function

$$\frac{2x^3 + x^2 + 4x + 1}{(x^2+1)^2}$$

as a sum of partial fractions.

Solution:

$$\frac{2x^3+x^2+4x+1}{(x^2+1)^2} = \frac{Ax+B}{(x^2+1)} + \frac{Cx+D}{(x^2+1)^2}$$

$$= \frac{(Ax+B)(x^2+1) + Cx + D}{(x^2+1)^2}$$

$$= \frac{Ax^3 + Bx^2 + (A+C)x + (B+D)}{(x^2+1)^2}$$

$\Rightarrow$ $A = 2$, $B = 1$, $C = 2$, $D = 0$

$$= \frac{2x+1}{x^2+1} + \frac{2x}{(x^2+1)^2}$$

(b) (10 points) Hence evaluate the integral

$$\int \frac{2x^3 + x^2 + 4x + 1}{(x^2+1)^2}\, dx$$

Solution:

$$\int \frac{2x^3+x^2+4x+1}{(x^2+1)^2}\, dx = \int \frac{2x}{x^2+1}\, dx + \int \frac{1}{x^2+1}\, dx + \int \frac{2x}{(x^2+1)^2}\, dx$$

$$= \ln|x^2+1| + \arctan(x) + \frac{-1}{(x^2+1)} + C$$

$u = x^2+1$ substitution

$u = x^2+1$ substitution

2. Evaluate the following improper integrals (if divergent, write divergent and explain your reasoning):

   (a) (10 points)

$$\int_0^{\frac{\pi}{4}} \frac{\sec(x)}{x^{\frac{3}{2}}} dx$$

(Hint: use the Comparison test)

Solution:

$\sec(x) \geqslant 1$ for all x in $[0, \frac{\pi}{4}]$

Hence $\frac{\sec(x)}{x^{\frac{3}{2}}} \geqslant \frac{1}{x^{\frac{3}{2}}}$ for all x in $(0, \frac{\pi}{4}]$.

$\frac{3}{2} > 1 \Rightarrow \int_0^{\frac{\pi}{4}} \frac{1}{x^{\frac{3}{2}}} dx$ divergent $\Rightarrow \int_0^{\frac{\pi}{4}} \frac{\sec(x)}{x^{\frac{3}{2}}} dx$ divergent.

   (b) (10 points)

$$\int_0^{\infty} \frac{x^3}{\sqrt{7+x^4}} dx$$

Substitute $u = 7 + x^4 \Rightarrow \frac{du}{dx} = 4x^3 \Rightarrow dx = \frac{1}{4x^3} du$

Hence $\int_0^{\infty} \frac{x^3}{\sqrt{7+x^4}} dx = \lim_{t \to \infty} \int_0^{t} \frac{x^3}{\sqrt{7+x^4}} dx = \lim_{t \to \infty} \int_7^{7+t^4} \frac{1}{4} (u)^{-\frac{1}{2}} du$

$= \lim_{t \to \infty} \left[\frac{1}{2} u^{\frac{1}{2}} \right]_7^{7+t^4} = \lim_{t \to \infty} \left(\frac{1}{2}(7+t^4)^{\frac{1}{2}} - \frac{1}{2} 7^{\frac{1}{2}} \right)$

$= \infty$

Hence $\int_0^{\infty} \frac{x^3}{\sqrt{7+x^4}} dx$ divergent.

PLEASE TURN OVER

4. (20 points) Find the arc length of the curve given by the function

$$y = x^2 - 2x + 6 - \frac{1}{8}\ln(x-1),$$

between $x = 2$ and $x = 4$.

Solution:

$$\text{Arc Length} = \int_2^4 \sqrt{1 + \left(2(x-1) - \frac{1}{8(x-1)}\right)^2}\, dx$$

Substitute $u = x - 1$, so $du = dx$. Then

$$\text{Arc Length} = \int_1^3 \sqrt{1 + \left(2u - \frac{1}{8u}\right)^2}\, du$$

$$= \int_1^3 \sqrt{\frac{64u^2 + 16^2u^4 - 32u^2 + 1}{64u^2}}\, du$$

$$= \int_1^3 \sqrt{\frac{16^2u^4 + 32u^2 + 1}{64u^2}}\, du$$

$$= \int_1^3 \sqrt{\left(\frac{16u^2 + 1}{8u}\right)^2}\, du$$

$$= \int_1^3 \frac{16u}{8} + \frac{1}{8u}\, du = \left[u^2 + \frac{1}{8}\ln|u|\right]_1^3 = 8 + \frac{1}{8}\ln(3)$$

5. (a) (10 points) Assume that $f(0) = 4$. Use the Midpoint Rule with $n = 5$ to approximate the value of $f(10)$ where $f'(x)$ takes the following values:

x	0	1	2	3	4	5	6	7	8	9	10
$f'(x)$	2	4	3	3	7	6	4	1	5	6	3

Solution:

$$\int_0^{10} f'(x)\,dx \approx M_5 = 2(4+3+6+1+6) = 40$$

$$f(10) - f(0) \Rightarrow f(10) \approx 40 + 4 = 44$$

(b) (10 points) Assuming that $|f^{(3)}(x)| \leq 1$, for all $0 \leq x \leq 10$, how large an n would you need to choose to guarantee that the above estimate is within 0.001 of the true value $f(10)$? You do not need to give an exact value, just a rough bound.

Solution:

Estimating $f(10)$ to within 0.001 is equivalent to estimating $\int_0^{10} f'(x)\,dx$ to within 0.001. (after adding 4).

Observe that $f'^{(2)}(x) = f^{(3)}(x)$. Hence choose $K=1$ in Midpoint Error Bound. Hence need n such that

$$\frac{10^3}{24n^2} < \frac{1}{1000} \Leftrightarrow \sqrt{\frac{1000000}{24}} < n$$

$$\Leftrightarrow \frac{1000}{\sqrt{24}} < n$$

END OF EXAM