

Lecture 8 : Improper Integrals

When we define definite integrals we need 2 things :

- 1) A finite, closed interval $[a, b]$
- 2) A continuous function $f(x)$ on $[a, b]$.

An improper integral is a definite integral where we relax these conditions.

Type 1 : Infinite Intervals

A) If $f(x)$ is a continuous function on $[a, \infty)$,

then define

$$\int_a^{\infty} f(x) dx = \lim_{t \rightarrow \infty} \left(\int_a^t f(x) dx \right)$$

B) If $f(x)$ is a continuous function on $(-\infty, b]$, then

define

$$\int_{-\infty}^b f(x) dx = \lim_{t \rightarrow -\infty} \left(\int_t^b f(x) dx \right)$$

If these limits exist we say the integral is convergent.

If not we say the integral divergent

C) If $f(x)$ is continuous on the whole number line, then

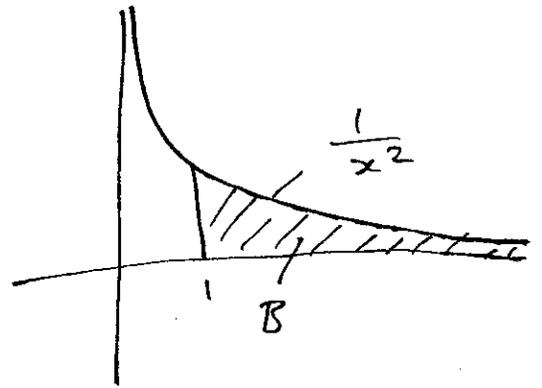
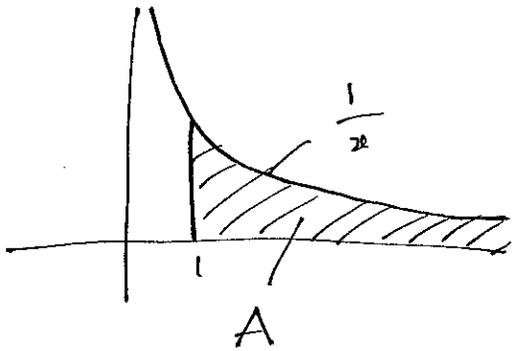
define

$$\int_{-\infty}^{\infty} f(x) dx = \int_0^{\infty} f(x) dx + \int_{-\infty}^0 f(x) dx$$

$\int_{-\infty}^{\infty} f(x) dx$ is said to be convergent if and only if
both $\int_0^{\infty} f(x) dx$ and $\int_{-\infty}^0 f(x) dx$ are convergent.

Examples $f(x) = \frac{1}{x^2}$ or $\frac{1}{x}$ on $[1, \infty)$

Pictures :



$$\begin{aligned} \text{Area (A)} &= \int_1^{\infty} \frac{1}{x} dx = \lim_{t \rightarrow \infty} \left(\int_1^t \frac{1}{x} dx \right) \\ &= \lim_{t \rightarrow \infty} \left([\ln(x)]_1^t \right) \\ &= \lim_{t \rightarrow \infty} (\ln(t)) \end{aligned}$$

This limit does not exist, hence integral is divergent
and A has infinite area.

$$\begin{aligned} \text{Area (B)} &= \int_1^{\infty} \frac{1}{x^2} dx = \lim_{t \rightarrow \infty} \left(\int_1^t \frac{1}{x^2} dx \right) \\ &= \lim_{t \rightarrow \infty} \left(\left[\frac{-1}{x} \right]_1^t \right) \\ &= \lim_{t \rightarrow \infty} \left(1 - \frac{1}{t} \right) = 1 \end{aligned}$$

Hence $\int_1^{\infty} \frac{1}{x^2} dx$ convergent and $\text{Area}(B) = 1$.

For more examples see example 2 on page 521 and example 3 on p 522.

Type 2

A) Assume $f(x)$ is a continuous function on $(a, b]$ with an infinite discontinuity at a , e.g.

$f(x) = \frac{1}{x}$ on $(0, 1]$, then define

$$\int_a^b f(x) dx = \lim_{t \rightarrow a^+} \left(\int_t^b f(x) dx \right)$$

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approaches
 a from above

B) Assume $f(x)$ is continuous on $[a, b)$ with an infinite discontinuity at b , then define

$$\int_a^b f(x) dx = \lim_{t \rightarrow b^-} \left(\int_a^t f(x) dx \right)$$

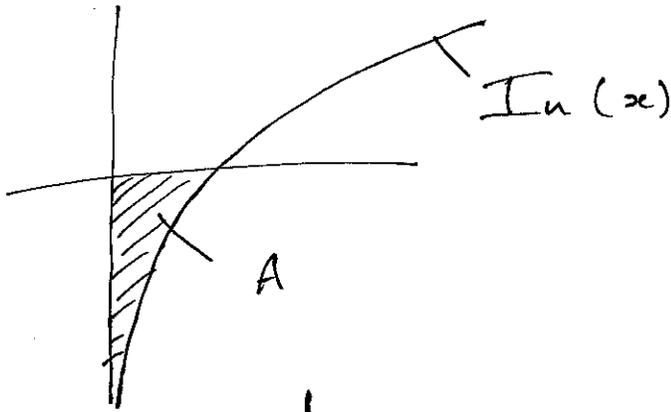
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approaches
 b from below

$\int_a^b f(x) dx$ is called convergent if the corresponding limit exists, and divergent otherwise.

c) If $f(x)$ has an infinite discontinuity at $a < c < b$, and both $\int_a^c f(x) dx$ and $\int_c^b f(x) dx$ are convergent, we define

$$\int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx.$$

Example $f(x) = \ln(x)$ on $(0, 1]$.



$$-\text{Area}(A) = \int_0^1 \ln(x) dx = \lim_{t \rightarrow 0^+} \left(\int_t^1 \ln(x) dx \right)$$

$$= \lim_{t \rightarrow 0^+} \left[x \ln(x) - x \right]_t^1$$

$$= \lim_{t \rightarrow 0^+} (-t \ln(t) - 1 + t)$$

$$\lim_{t \rightarrow 0^+} (t \ln(t)) = \lim_{t \rightarrow 0^+} \left(\frac{\ln(t)}{1/t} \right) \underset{\uparrow}{=} \lim_{t \rightarrow 0^+} \left(\frac{1/t}{-1/t^2} \right)$$

$$\text{L'Hopital} = \lim_{t \rightarrow 0^+} (-t) = 0$$

Hence $\int_0^1 \ln(x) dx$ convergent and $\text{Area}(A) = 1$.

For more examples see Examples 6 and 7 on page 524.

Clever Trick : (The comparison test)

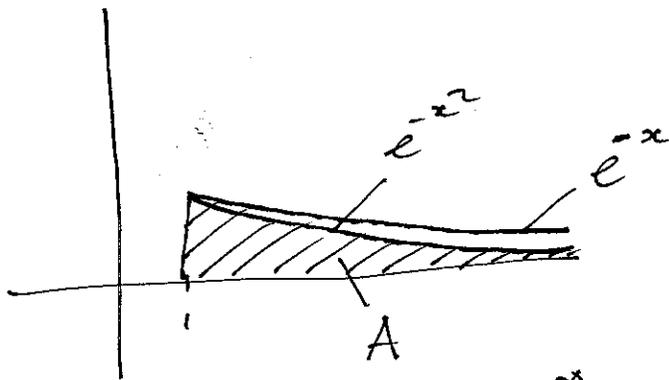
Consider $\int_1^{\infty} e^{-x^2} dx = \lim_{t \rightarrow \infty} \left(\int_1^t e^{-x^2} dx \right)$

problem : can't work out $\int_1^t e^{-x^2} dx$!

Notice, though, that ~~for~~ for all $x \geq 1$

$$x^2 \geq x \Rightarrow 0 \leq e^{-x^2} \leq e^{-x}$$

Picture :



Hence $\text{Area}(A) = \int_1^{\infty} e^{-x^2} dx \leq \int_1^{\infty} e^{-x} dx$.

But $\int_1^{\infty} e^{-x} dx = \lim_{t \rightarrow \infty} \left(\int_1^t e^{-x} dx \right) = \lim_{t \rightarrow \infty} \left([-e^{-x}]_1^t \right)$

$$= \lim_{t \rightarrow +\infty} (e^{-1} - e^{-t}) = e^{-1}$$

Hence $\int_1^{\infty} e^{-x^2} dx$ is bounded above by e^{-1} so

must be finite and thus convergent! Awesome!

This gives us a method to show if an improper integral is convergent. The problem is it doesn't help us to find the actual answer. Also it requires spotting a clever choice of bounding function, whose improper integral we can calculate.

We could also use this method to show an improper integral is divergent. e.g. For $x > 1$

$$0 \leq \frac{1}{x} \leq \frac{\sin(x) + 2}{x}$$

$$\Rightarrow \int_1^{\infty} \frac{1}{x} dx \leq \int_1^{\infty} \frac{\sin(x) + 2}{x} dx \Rightarrow \int_1^{\infty} \frac{\sin(x) + 2}{x} dx$$

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 if convergent must diverge.

For a precise statement of comparison test see page 525.