

Lecture 5 : Integration of rational functions by partial fractions.

A polynomial is a function of the form :

$$P(x) = a_n x^n + \dots + a_1 x + a_0,$$

where n is a non-negative integer and each a_i is a real number.

If $a_n \neq 0$ we say the degree of $P(x)$ is n , denoted $\deg(P(x)) = n$.

e.g. Quadratic polynomial = polynomial of degree 2.

We say a function $f(x)$ is rational if $f(x) = \frac{P(x)}{Q(x)}$,

where $P(x)$ and $Q(x)$ are polynomials.

Today we will solve integrals of the form $\int \frac{P(x)}{Q(x)} dx$.

Important Examples : $\int x^n dx = \begin{cases} \frac{1}{n+1} x^{n+1} + C & (n \neq -1) \\ \ln|x| + C & (n = -1) \end{cases}$

$$\int \frac{1}{x^2+1} dx = \arctan(x) + C, \quad \int \frac{2x}{x^2+1} dx = \ln|x^2+1| + C.$$

Strategy Express $\frac{P(x)}{Q(x)}$ as a sum of more basic rational functions (called partial fractions) whose integrals we can solve.

Basic Example :
$$\begin{aligned} \int \frac{2}{x^2-1} dx &= \int \frac{1}{x-1} - \frac{1}{x+1} dx \\ &= \ln|x-1| - \ln|x+1| + C \\ &= \ln\left(\frac{|x-1|}{|x+1|}\right) + C \end{aligned}$$

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Step 1 If $\deg(P(x)) > \deg(Q(x))$ divide $P(x)$ by $Q(x)$, using long division, until a remainder polynomial $R(x)$

is obtained, such that $\deg(R(x)) < \deg(Q(x))$ and

$$\frac{P(x)}{Q(x)} = S(x) + \frac{R(x)}{Q(x)}$$

 (Note: If $\deg(P(x)) \leq \deg(Q(x))$
 then $S(x) = 0$ and
 $P(x) = R(x)$)
 polynomial.

Example :
$$\int \frac{2x^5 + x^3 + 2x^2 + x + 2}{x^3 + 1} dx = (2x^2 + 1) + \frac{x+1}{x^3 + 1}$$

Step 2
$$\int \frac{P(x)}{Q(x)} dx = \int S(x) dx + \int \frac{R(x)}{Q(x)} dx$$

easy just
 polynomial integration.

Focus on $\int \frac{R(x)}{Q(x)} dx$.

Now factor $Q(x)$ into linear and quadratic terms as much as possible.

Example : $x^3 - 1 = (x-1)(x^2 + x + 1)$
 no real roots so cannot factor further.

In general $Q(x)$ will factor into terms of the form

$$\underline{\underline{ax+b}} \quad \text{and} \quad \underline{\underline{ax^2+bx+c}}, \text{ where } b^2 - 4ac < 0$$

Now gather the repeated linear and quadratic factors.

Example : $x^6 + 2x^4 + x^2 = x^2(x^2 + 1)^2$

Step 3 Express $\frac{R(x)}{Q(x)}$ as a sum of rational functions
 of the following forms :

$$\frac{A}{(ax+b)^i}, \quad 1 \leq i \leq r \quad \text{and} \quad \frac{Ax+B}{(ax^2+bx+c)^j}, \quad 1 \leq j \leq s,$$

where $(ax+b)^i$ and $(ax^2+bx+c)^j$ are factors of $Q(x)$
obtained in Step 2.

Must solve for
 $\downarrow A, B, C$.

Example

$$\begin{aligned}\frac{4x^2 - 8x + 2}{x(x-1)(x-2)} &= \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x-2} \\ &= \frac{A(x-1)(x-2) + Bx(x-2) + Cx(x-1)}{x(x-1)(x-2)} \\ &= \frac{(A+B+C)x^2 + (-3A-2B-C)x + (2A)}{x(x-1)(x-2)}\end{aligned}$$

$$\begin{aligned}\Rightarrow 4 &= A+B+C \\ -8 &= -3A-2B-C \\ 2 &= 2A\end{aligned} \Rightarrow A = 1, B = 2, C = 1$$

More general : $\frac{x^3+x+3}{(x^2+1)^3(x-1)^2} = \frac{A}{(x-1)} + \frac{B}{(x-1)^2} + \frac{Cx+D}{(x^2+1)} + \frac{Ex+F}{(x^2+1)^2} + \frac{Gx+H}{(x^2+1)^3}$

In this case we do same as above and end up with a collection of linear equations in A, B, C, D, E, F, G, H . We could solve these using linear algebra or maybe a computer.

Step 4 Integrate each term in partial sum using techniques we know.

Examples

$$1) \int \frac{2}{3x+4} dx = \frac{2}{3} \ln |3x+4| + C$$



$$2) \int \frac{7}{(6x+1)^3} dx = \frac{7}{-2} \cdot (6x+1)^{-2} + C$$

$$= \frac{-7}{12(6x+1)^2} + C$$

$$3) \int \frac{2x+3}{(x^2+2x+2)^2} dx = \int \frac{2(x+1)+1}{((x+1)^2+1)^2} dx$$

complete the square.

Change variable to $u = x+1 \Rightarrow \frac{du}{dx} = 1 \Rightarrow du = dx$

$$\Rightarrow \int \frac{2x+3}{(x^2+2x+2)^2} dx = \int \frac{2u+1}{(u^2+1)^2} du = \int \frac{2u}{(u^2+1)^2} du + \int \frac{1}{(u^2+1)^2} du$$

$$= \frac{-1}{u^2+1} + \int \frac{1}{(u^2+1)^2} du$$

Move difficult.

use usual

$v = u^2$ substitution

Clever trick: Evaluate $\int \frac{1}{u^2+1} du$ using integration by

parts, with $f(u) = \frac{1}{u^2+1}$, $g'(u) = 1$, $g(u) = u$.

$$\Rightarrow \int \frac{1}{u^2+1} du = \frac{u}{u^2+1} - \int \frac{-2u^2}{(u^2+1)^2} du = \frac{u}{u^2+1} + 2 \int \frac{(u^2+1)-1}{(u^2+1)^2} du$$

$$= \frac{u}{u^2+1} + 2 \int \frac{1}{u^2+1} du - 2 \int \frac{1}{(u^2+1)^2} du$$

$$\Rightarrow \int \frac{1}{(u^2+1)^2} du = \frac{1}{2} \left(\frac{u}{u^2+1} + \int \frac{1}{u^2+1} du \right)$$

$$= \frac{u}{2(u^2+1)} + \frac{1}{2} \arctan(u) + C$$

In conclusion :

$$\int \frac{2x+3}{(x^2+2x+2)^2} dx = \frac{x+1}{2((x+1)^2+1)} + \frac{1}{2} \arctan(x+1) + C.$$

Using this type of trick repeatedly, along with some clever trigonometric substitutions we can solve more complicated examples,

for example :

$$\int \frac{4x+3}{(x^2+4x+5)^6} dx.$$

Conclusion : Putting all this together we can always solve an integral at a rational function. If the degrees of the repeated factors in $Q(x)$ are very big it could be very time consuming though.