

Lecture 38 : Non homogeneous 2nd Order Linear Equations
(continued)

We're trying to find particular solutions to differential equations of the form

$$a \frac{d^2y}{dx^2} + b \frac{dy}{dx} + cy = G(x),$$

where a, b, c are constants and $G(x)$ is a continuous function.

Last lecture we observed the following :

1/ If $G(x)$ is a polynomial try $y_p(x)$ a polynomial of same degree. If $c=0$, try $y_p(x)$ one degree higher. If $c=0$ and $b=0$, try $y_p(x)$ two degrees higher. E.g. $\frac{d^2y}{dx^2} + \frac{dy}{dx} = 2 + 2x$ has particular solution $y_p(x) = x^2$.

2/ If $G(x) = e^{kx}$, try $y_p(x) = Ae^{kx}$.

3/ If $G(x) = (\cos(kx))$ or $(\sin(kx))$ try $y_p(x) = A \cos(kx) + B \sin(kx)$.

If $G(x)$ is a product of functions of the above type, then we take as a trial solution a product of the appropriate trial solutions. For example, to solve

$$y'' - 4y = xe^x$$

we would try a trial solution

$$y_p(x) = (Ax + B)e^x$$

(Note that the unknown constant in front of e^x in case 2/ is absorbed by A and B)

As before we first evaluate the left hand side and reduce the equality to a system of linear equations which we then solve. In the above example

$$y_p'' - 4y_p = (-3Ax + 2A - 3B)e^x$$

Hence to find a solution must solve

$$-3A = 1 \Rightarrow A = \frac{-1}{3}$$

$$2A - 3B = 0 \quad B = \frac{2}{9}$$

Hence $y_p(x) = \left(\frac{-1}{3}x - \frac{2}{9}\right)e^x$ is a particular solution.

If we have 2 differential equations

$$ay'' + by' + cy = G_1(x) \text{ and } ay'' + by' + cy = G_2(x)$$

with particular solutions $y_{p_1}(x)$ and $y_{p_2}(x)$ respectively

then it is straightforward to check that $y_{p_1}(x) + y_{p_2}(x)$ must be a solution to

$$ay'' + by' + cy = G_1(x) + G_2(x).$$

Hence if $G(x)$ is a sum of functions of the preceding, then we take as a trial solution a sum of the appropriate trial solutions. For example, to find a particular ~~trial~~ solution to

$$y'' - 4y = xe^x + \cos(2x)$$

first find $y_{p_1}(x)$ a particular solution to

$$y'' - 4y = xe^x.$$

Then find $y_{p_2}(x)$ a particular solution to

$$y'' - 4y = \cos(2x)$$

Then take their sum.

We've already found $y_{p_1}(x) = \left(\frac{-1}{3}x - \frac{2}{9}\right)e^x$

To determine a $y_{p_2}(x)$, try

$$y_{p_2}(x) = C \cos(2x) + D \sin(2x)$$

Then $y_{p_2}'' - 4y_{p_2} = -8 \cos(2x) - 8D \sin(2x)$

Hence must solve $\begin{aligned} -8C &= 1 \\ -8D &= 0 \end{aligned} \Rightarrow \begin{aligned} C &= \frac{-1}{8} \\ D &= 0 \end{aligned}$

$$\Rightarrow y_{p_2}(x) = \frac{-1}{8} \cos(2x)$$

Thus a particular solution is

$$y_p(x) = y_{p_1}(x) + y_{p_2}(x) = \left(\frac{-1}{3}x - \frac{2}{9}\right)e^x - \frac{1}{8} \cos(2x)$$

The only problem we might run into in these strategies is that our trial solution may actually be a solution to the homogeneous complementary equation. For example

$$y'' + 2y' - 3y = e^x$$

In such cases we multiply the recommended trial solution by x (or by x^2 if necessary) so that no terms in $y_p(x)$

are solutions to the complementary equation. In the above case we should try

$$y_p(x) = Axe^x.$$

If we were given $y'' - 2y' + y = e^x$ we would observe that both e^x and xe^x are solutions to

$$y'' - 2y' + y = 0$$

Thus we should try

$$y_p(x) = Ax^2e^x.$$

For a more complicated example look at example 6 on page 1153