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Lecture 37 : Nonhomogeneous 2<sup>nd</sup>-Order Linear Differential Equations


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Today we will study solutions to nonhomogeneous linear equations of the form

$$a \frac{dy^2}{dx^2} + b \frac{dy}{dx} + cy = G(x),$$

where  $a, b, c$  are constants and  $G(x)$  is a not-identically-zero function. Associated to any such nonhomogeneous equation is the homogeneous equation

$$a \frac{dy^2}{dx^2} + b \frac{dy}{dx} + cy = 0.$$

We call this the complementary equation.

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Fact : The general solution of the above nonhomogeneous equation can be written as

$$y(x) = y_p(x) + y_c(x),$$

where  $y_p(x)$  is a fixed particular solution and  $y_c(x)$  is the general solution to the complementary equation.

For a proof of this fact see page 1149. It's not hard.

Example Solve  $\frac{d^2y}{dx^2} - 3 \frac{dy}{dx} + 2y = 2x - 3$ .

First find general solution to  $\frac{d^2y}{dx^2} - 3 \frac{dy}{dx} + 2y = 0$ .

Find roots of  $r^2 - 3r + 2 = (r-1)(r-2)$ . Hence

$$y_c(x) = c_1 e^{2x} + c_2 e^x,$$

for constants  $c_1$  and  $c_2$ . Note that  $y_p(x) = x$  is a particular solution. Hence a general solution is

$$y(x) = x + c_1 e^{2x} + c_2 e^x.$$

Basic Strategy : 1/ Find general solution to homogeneous complementary equation, using techniques from last week.  
 2/ Depending on  $G(x)$  try different possible particular solutions, until we find one.

Case 1  $G(x)$  is a polynomial in  $x$ . In this case look for a solution  $y_p(x)$  which is a polynomial of the same degree.

Example :  $\frac{d^2y}{dx^2} + \frac{dy}{dx} - 2y = x^2$

Look for a solution of the form  $y_p(x) = Ax^2 + Bx + C$ . ↑ degree 2 polynomial

$$\frac{dy_p}{dx} = 2Ax + B$$

$$\frac{d^2y_p}{dx^2} = 2A$$

Hence  $\frac{d^2y_p}{dx^2} + \frac{dy_p}{dx} - 2y_p = 2A + 2Ax + B - 2(Ax^2 + Bx + C)$

$$= (-2A)x^2 + (2A - 2B)x + (2A + B - 2C)$$

Need  $A, B, C$  such that this equals  $x^2$ . Hence

$$\begin{aligned} -2A &= 1 \\ 2A - 2B &= 0 \\ 2A + B - 2C &= 0 \end{aligned} \Rightarrow \begin{aligned} A &= \frac{-1}{2} \\ B &= \frac{-1}{2} \\ C &= \frac{-3}{4} \end{aligned}$$

Hence  $y_p(x) = \frac{-1}{2}x^2 - \frac{1}{2}x - \frac{3}{4}$  is a particular solution.

Case 2  $y(x) = Ce^{kx}$  for constants  $k$  and  $C$ .

In this case try a particular solution of the form

$$y_p(x) = Ae^{kx}$$

Example  $y'' + 4y = e^{3x}$ . We seek a particular solution

of the form  $y_p(x) = Ae^{3x}$ .

$$y_p''(x) = 9Ae^{3x} \Rightarrow y_p'' + 4y_p = (9A + 4A)e^{3x} = 13Ae^{3x}$$

Hence need  $13A = 1 \Rightarrow A = \frac{1}{13}$ . Thus  $y_p(x) = \frac{1}{13}e^{3x}$

is a particular solution.

Case 3 If  $G(x)$  is of the form  $(\cos(kx))$  or  $(\sin(kx))$

then try  $y_p(x) = A \cos(kx) + B \sin(kx)$

Example  $y'' + y' - 2y = \sin(x)$ . We seek a particular solution

of the form  $y_p(x) = A \cos(x) + B \sin(x)$

$$y_p' = -A \sin(x) + B \cos(x)$$

$$y_p'' = -A \cos(x) - B \sin(x)$$

$$\begin{aligned} \text{Hence } y_p'' + y_p' - 2y_p &= (-A \cos(x) - B \sin(x)) + (-A \sin(x) + B \cos(x)) \\ &\quad - 2(A \cos(x) + B \sin(x)) \\ &= (-3A + B) \cos(x) + (-A - 3B) \sin(x). \end{aligned}$$

Need  $A$  and  $B$  such that this equals  $\sin(x)$ . Hence

$$\begin{aligned} -3A + B &= 0 \\ -A - 3B &= 1 \end{aligned} \Rightarrow \begin{aligned} A &= \frac{-1}{10} \\ B &= \frac{3}{10} \end{aligned}$$

Thus  $y_p(x) = \frac{-1}{10} \cos(x) - \frac{3}{10} \sin(x)$  is a particular solution.

In all these cases to find a general solution we would also need to find a general solution to the homogeneous complementary equation.