

Lecture 36 : 2nd order homogeneous linear differential equations (continued)

After having introduced the complex numbers let's take a look back at case 3. Recall this is when we are trying to solve

$$a \frac{d^2y}{dx^2} + b \frac{dy}{dx} + cy = 0$$

and $b^2 - 4ac < 0$. This assumption means that

$ar^2 + br + c = 0$ has no real solutions. However, it has two solutions in the complex numbers, namely

$$\begin{aligned} \text{e.g. } \sqrt{-3} &= \sqrt{3}i \\ &= \sqrt{3}i \end{aligned}$$

$$r_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} = \frac{-b}{2a} + \left(\frac{\sqrt{4ac - b^2}}{2a} \right) i$$

$$r_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a} = \frac{-b}{2a} - \left(\frac{\sqrt{4ac - b^2}}{2a} \right) i$$

As in Lecture 34, let $\alpha = \frac{-b}{2a}$ and $\beta = \frac{\sqrt{4ac - b^2}}{2a}$

Thus by same logic as in Lecture 34 a general solution is given by $y(x) = c_1 e^{(\alpha + \beta i)x} + c_2 e^{(\alpha - \beta i)x}$

where c_1 and c_2 are constants (they could be complex now). Let's simplify this with Euler's Formula:

$$\begin{aligned}
 y(x) &= c_1 e^{\alpha x} e^{i\beta x} + c_2 e^{\alpha x} e^{-i\beta x} \\
 &= e^{\alpha x} (c_1 (\cos \beta x + i \sin \beta x) + c_2 (\cos \beta x - i \sin \beta x)) \\
 &= e^{\alpha x} ((c_1 + c_2) \cos \beta x + i(c_1 - c_2) \sin \beta x)
 \end{aligned}$$

Let $C_1 = c_1 + c_2$ and $C_2 = i(c_1 - c_2)$

We deduce that a general solution (with real functions)

is $y(x) = C_1 e^{\alpha x} \cos \beta x + C_2 e^{\alpha x} \sin \beta x$, where

C_1 and C_2 are real constants. This is where

$e^{\alpha x} \cos \beta x$ and $e^{\alpha x} \sin \beta x$ come from in case 3 from
lecture 34.

Initial and Boundary value problems

For a first-order differential equation an initial condition is the requirement that a solution satisfies $y(x_0) = y_0$ for some fixed values x_0 and y_0 .

The general solution to a first-order differential equation only ever had one unknown constant, e.g.

$\frac{dy}{dx} = y$ has general solution $y(x) = Ae^x$ for A any constant, thus demand $y(x_0) = y_0$ fixes a unique value of A and thus a unique solution.

As we have seen, general solutions to 2nd-order differential equations involve 2 unknown constants c_1 and c_2 .

c_2 , thus to get a unique solution we must give 2 initial conditions. This means

$$y(x_0) = y_0 \quad \text{and} \quad y'(x_0) = y_1$$

for constants x_0, y_0 and y_1 . These requirements give unique values for c_1 and c_2 and thus a unique solution. Let's do an example.

Solve : $y'' + y' - 6y = 0$ with $y(0) = 1, y'(0) = 0$

$$\text{Find roots of } \cancel{r^2 + 3r - 10} \quad r^2 + r - 6 = 0$$

$$\begin{aligned} &|| \\ &(r+3)(r-2) \end{aligned}$$

Thus $r_1 = 2$ and $r_2 = -3 \Rightarrow$ general solution is

$$y(x) = C_1 e^{2x} + C_2 e^{-3x}$$

$$y(0) = 1 \Rightarrow C_1 e^{2 \cdot 0} + C_2 e^{-3 \cdot 0} = C_1 + C_2 = 1$$

$$y'(x) = 2C_1 e^{2x} + (-3C_2) e^{-3x} \Rightarrow$$

$$y'(0) = 2C_1 e^{2 \cdot 0} - 3C_2 e^{-3 \cdot 0} = 2C_1 - 3C_2 = 0$$

So we must find simultaneous solutions to

$$C_1 + C_2 = 1 \quad (1)$$

$$2C_1 - 3C_2 = 0 \quad (2)$$

$$(1) \Rightarrow C_2 = 1 - C_1$$

$$\begin{aligned} (2) \Rightarrow 2C_1 - 3(1 - C_1) &= 0 \Rightarrow 2C_1 + 3C_1 - 3 = 0 \\ &\Rightarrow 5C_1 - 3 = 0 \Rightarrow C_1 = \frac{3}{5} \end{aligned}$$

$$\Rightarrow C_2 = \frac{2}{5}$$

Hence unique solution is $y(x) = \frac{3}{5} e^{2x} + \frac{2}{5} e^{-3x}$.

Conclusion To solve an initial value problem, find general solution, differentiate, and feed in initial condition and find simultaneous solutions.

The reason this works is because the general solution has 2 degrees of freedom (any choice of C_1 and C_2) and the initial conditions give 2 independent constraints.

Another way to give 2 constraints is boundary conditions, that is demanding

$$y(x_0) = y_0 \quad \text{and} \quad y(x_1) = y_1,$$

for some constants x_0, x_1, y_0, y_1 . ($x_0 \neq x_1$).

In this case, though, there may be no solutions.

Let's do an example:

Solve the boundary problem: $y'' + 2y' + y = 0$

subject to $y(0) = 1, y(1) = 3$.

As above must find roots of $r^2 + 2r + 1 = 0$

This has -1 as a repeated root (Case 2), thus a general solution is

$$y(x) = C_1 e^{-x} + C_2 x e^{-x}$$

Thus we need to find c_1 and c_2 such that

$$y(0) = c_1 = 1$$

$$y(1) = c_1 e^{-1} + c_2 e^{-1} = 3$$

$$\Rightarrow c_1 = 1 \quad \text{and} \quad c_2 = \frac{3 - e^{-1}}{e^{-1}} = 3e - 1$$

Hence the unique solution is

$$y(x) = e^{-x} + (3e - 1)x e^{-x}$$

