

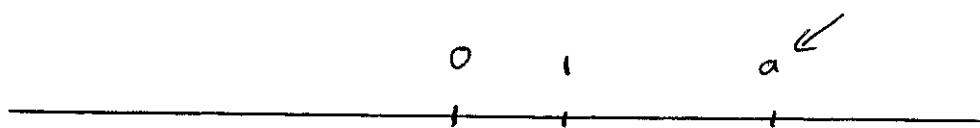
Math 1B: Calculus (Fall 2014)

Lecture 35 : Introduction to the Complex Numbers

What are numbers and what properties do they have?

For us a number is simply a point on the Number Line

the number a .



For example when we graph functions the x -axis is just a copy of the number line. Notice that any number is given by 2 pieces of data:

- 1) Its absolute value, $|a|$, that is the distance to 0 from a .
- 2) The sign of a , i.e. is a negative or positive.

So really a number is given by a magnitude (or absolute value) together with a direction (left for negative and right for positive).

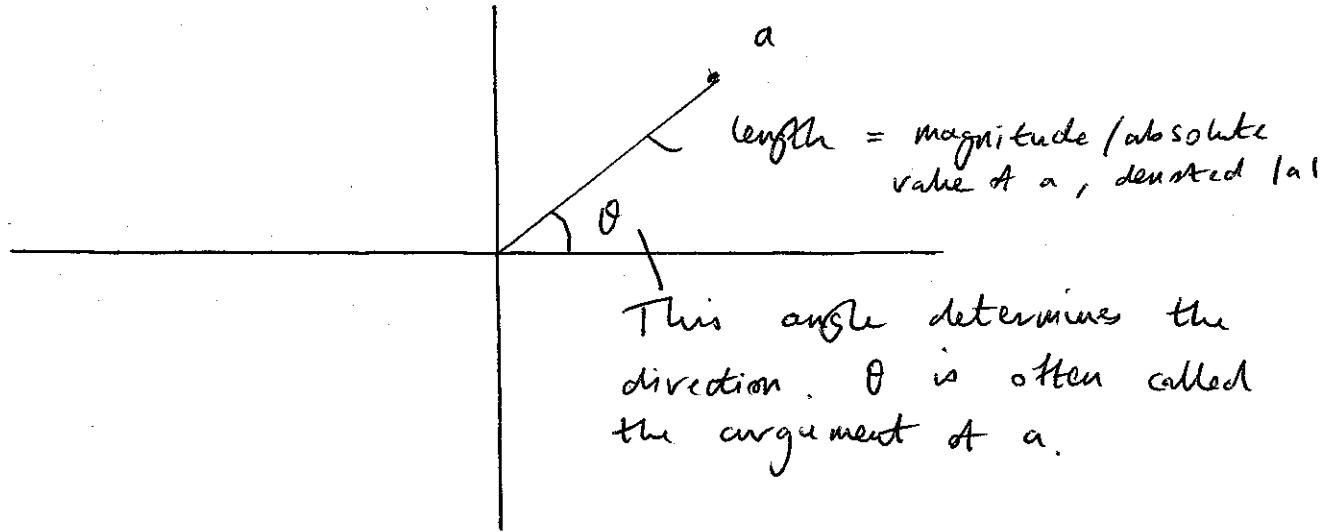
From this perspective there is a very natural generalisation of the concept of a number.

Definition

A complex number is a point in the plane.

(thought of as the X-Y plane if you like).

This is a generalisation because, using polar coordinates, a point in the plane is just given by a magnitude (distance from the origin) and a direction:

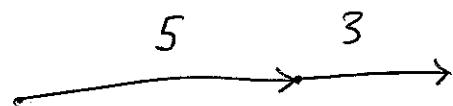


Observe that the ordinary number line sits inside the complex numbers as the x-axis, because these points correspond to directions left and right from before.
(we identify the decimal number a with the point $(a, 0)$)
So all we've really done is expand the meaning of the word number from points on a line to points on a plane.

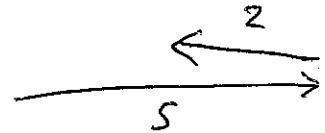
This all seems pretty silly. The real magic happens when we ask the following:

Can we define addition and multiplication on the complex numbers which extends the usual addition and multiplication on the number line?

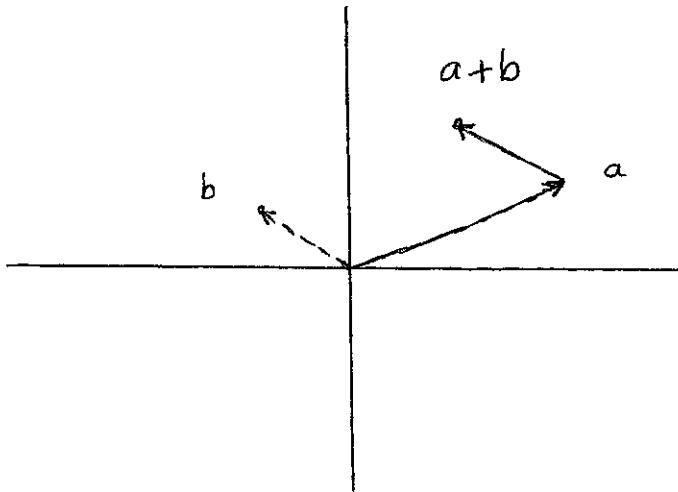
1) Addition. How do we add in the number line. For example what is $5 + 3$? We start at the origin (that's just 0 on the line) travel to the right 5 units, then travel the right 3 more units.



If it was ~~$5 + (-2)$~~ $5 + (-2)$ we would travel to right 5 units and then to the left 2 units.



This makes it obvious how to make sense of addition of complex numbers: Just take the arrows from the origin corresponding to a and b , lay them out tail to tip and see where we end up.



In cartesian coordinates this is really easy :

If $a = (x_1, y_1)$ and $b = (x_2, y_2)$ then

$$a+b = (x_1 + x_2, y_1 + y_2)$$

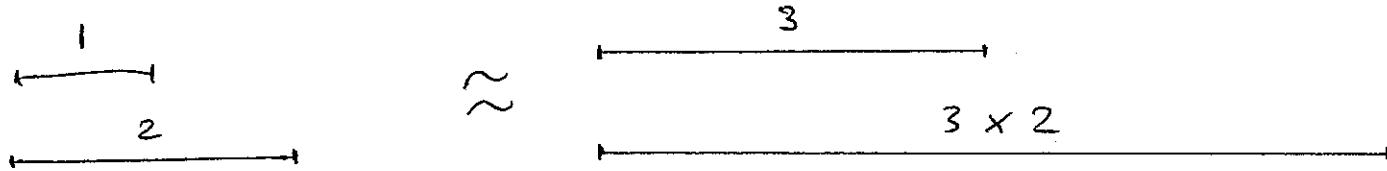
Even better, this addition behaves exactly like ordinary addition, e.g. $a+b = b+a$.

2) Multiplication. How can we come up with

a good concept of multiplying 2 complex numbers?

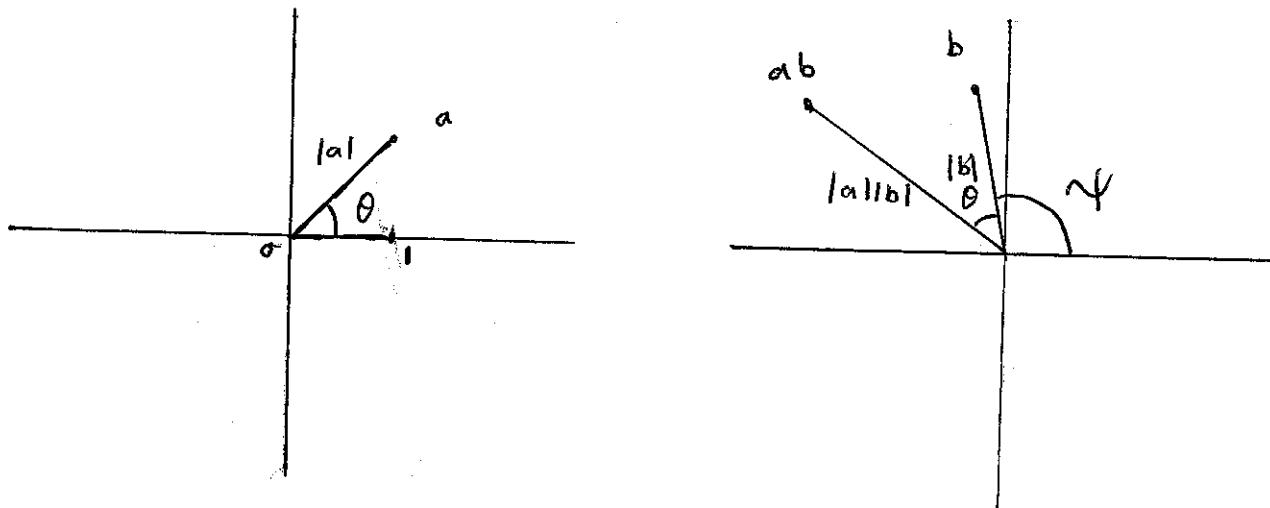
Let's carefully think about multiplication on the number line.

For example, what is 2×3 ? We can think about it in terms of ratios as follows:



i.e. The ratio of 1 to 2 is the same as the ratio of 3 to 3×2 .

Let's use this observation to ~~miss~~ generalise multiplication to the whole plane, that is the complex numbers.



Thus in polar coordinates if a is given by $(|a|, \theta)$ and b is given by $(|b|, \psi)$, then

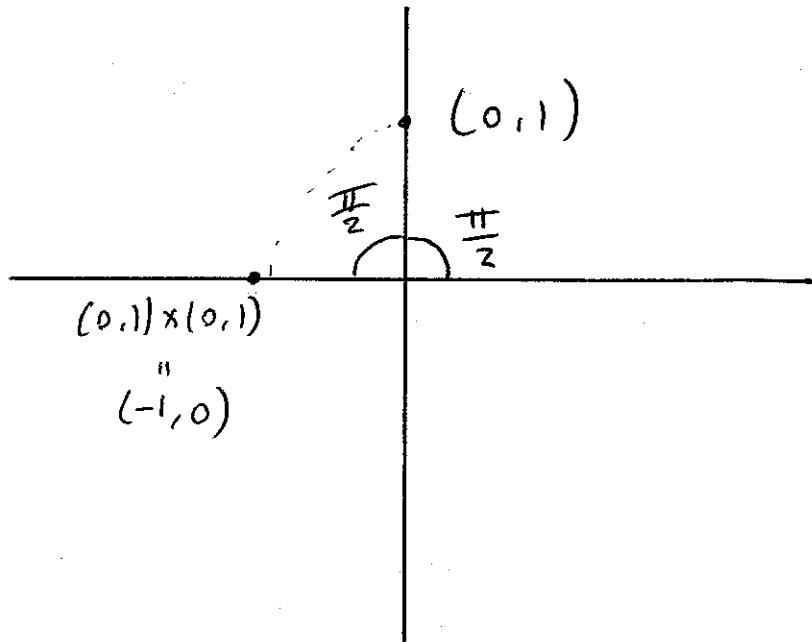
$$ab \text{ is given by } (|a||b| \cdot \psi + \theta)$$

i.e. multiplication is given by multiplying magnitudes and adding arguments.

This multiplication behaves exactly like the usual one, for example $(a+b)c = ac+bc$ for any a, b, c in the complex numbers.

So we've successfully generalised addition and multiplication from the line to the plane. Awesome!

Who cares? Why is this useful? Actually it's very very useful. For example consider the complex number given in cartesian coordinates $(0, 1)$. This is not an ordinary number because it does not lie on the x -axis (the usual number line). What is $(0, 1)$ in polar coordinates? It has magnitude 1 and argument $\frac{\pi}{2}$. Thus according to our class multiplication $(0, 1) \times (0, 1)$ is given by the point at magnitude 1×1 and argument $\frac{\pi}{2} + \frac{\pi}{2} = \pi$. This is just $(-1, 0)$ in cartesian coordinates, which is -1 in the usual notation. Here's a picture:



Written another way $(0,1)^2 = -1$. The point $(0,1)$ is so important we give it a name, we call it i . So we've just shown that $i^2 = -1$. Wow, so on the ordinary number line the polynomial $x^2 + 1$ has no zeros, but if we extend the number line to the plane using the complex numbers we do get a zero! This is a special case of the truly amazing fact:

The Fundamental Theorem of Algebra

Every non-constant polynomial has a zero in the complex numbers.

This was first proven by Gauss when he was 21.

Before we observe why this is true for quadratic polynomials, let's fix some notation: We write the point (x,y) as $x+iy$.

This makes sense because x really means $(x,0)$ and iy really means $(0,y)$.

So every point in the complex numbers can be uniquely written in the form $x+iy$.

Because addition and multiplication on the complex numbers obey the same rules as ordinary addition and multiplication we know:

$$(x_1 + iy_1)(x_2 + iy_2)$$

$$= x_1x_2 + iy_1x_2 + iy_2x_1 + i^2y_1y_2$$

$$= (x_1x_2 - y_1y_2) + (x_1y_2 + y_1x_2)i$$

This gives us a way of computing products using cartesian coordinates.

Remark People often call things like i imaginary.
This ~~suggest it has no meaning~~. It is just the point $(0,1)$ in the plane. People sometimes introduce the complex numbers by defining $i = \sqrt{-1}$. This fundamentally misunderstands the point. We defined the complex numbers by generalising addition and multiplication to the plane from the line, then observed the point $(0,1)$, labelled i , has square -1 . This is all completely logical. Defining $i = \sqrt{-1}$ makes it nothing more than a formal symbol without sensible meaning.

Let's try and understand zeros of quadratics in complex numbers.

Consider $x^2 + x + 1 = 0$. Here $a = b = c = 1 \Rightarrow$

$$b^2 - 4ac = -3$$

Thus the quadratic formula gives

$$r_1 = \frac{-1 + \sqrt{-3}}{2} \quad r_2 = \frac{-1 + \sqrt{-3}}{2}$$

$\sqrt{-3}$ doesn't make sense on the usual number line, however it is just the point $\sqrt{3} \cdot \sqrt{-1} = \sqrt{3}i$ in the complex plane. Hence $x^2 + x + 1 = 0$ has two solutions in the complex numbers

$$r_1 = \frac{-1}{2} + \frac{\sqrt{3}}{2}i \quad \text{and} \quad r_2 = \frac{-1}{2} - \frac{\sqrt{3}}{2}i$$

This pattern is followed whenever $b^2 - 4a < 0$.

Fantastic!

The final aspect of the complex numbers I want to discuss is exponentials. In particular what is the meaning of e^a where a is a complex number?

Let $a = x + iy$

Then

$$e^a = e^{x+iy} = e^x \cdot e^{iy}$$

↑ ↑
makes don't understand yet.
sense

Let's use the Maclaurin series for the exponential function.

Recall that for any decimal z we had.

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \dots$$

Fact: The right hand side is convergent for z any complex number. (The concept of a convergent series at complex numbers is basically identical to the usual one)

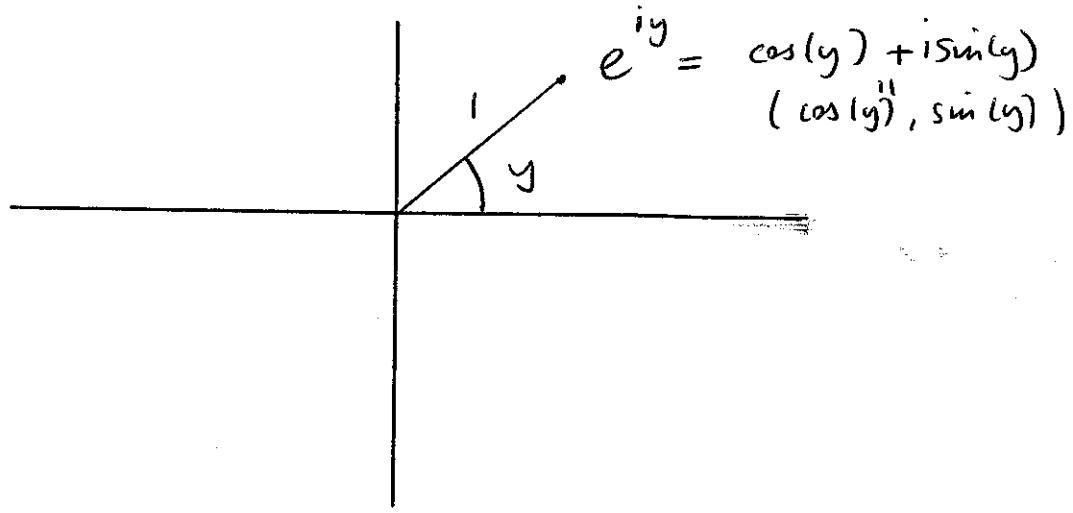
Hence

$$\begin{aligned} e^{iy} &= 1 + iy + \frac{(iy)^2}{2!} + \frac{(iy)^3}{3!} + \dots \\ &\stackrel{(i^2=-1)}{\text{remember}} = \left(1 - \frac{y^2}{2!} + \frac{y^4}{4!} \dots\right) + \left(y - \frac{y^3}{3!} + \frac{y^5}{5!} \dots\right)i \\ &= \cos(y) + i \sin(y) \quad (\text{Remember the Maclaurin series for cos and sin}) \end{aligned}$$

Hence we get the awesome formula

$$e^{iy} = \cos(y) + i \sin(y)$$

Note that this is just the point in the plane with magnitude 1 and argument y . i.e.



Finally observe that if $y = \pi$ then we get Euler's ~~most~~ celebrated formula

$$e^{i\pi} = -1$$

Written another way this is just

$e^{i\pi} + 1 = 0$

Thus it unites $e, i, \pi, 1$ and 0 in one ~~beautiful~~ beautiful equation.