

Lecture 34 Second-Order Linear Equations

A 2nd order linear differential equation has the form

$$P(x) \frac{d^2y}{dx^2} + Q(x) \frac{dy}{dx} + R(x)y = G(x),$$

where P, Q, R and G are continuous functions.

If $G(x) = 0$ for all x then we call the equation homogeneous. If $G(x) \neq 0$ for some x we call it

nonhomogeneous. Today we will focus on homogeneous equations.

Fact: If $y_1(x)$ and $y_2(x)$ are both solutions to the homogeneous equation:

$$P(x) \frac{d^2y}{dx^2} + Q(x) \frac{dy}{dx} + R(x)y = 0,$$

then for any constant c_1 and c_2 the function

$$y(x) = c_1 y_1(x) + c_2 y_2(x)$$

is also a solution.

This easily follows from the fact that

$$\frac{d(c_1 y_1 + c_2 y_2)}{dx} = c_1 \frac{dy_1}{dx} + c_2 \frac{dy_2}{dx}$$

Remember that c_1 and c_2 are constant numbers here.

Definition We say that two functions $y_1(x)$ and $y_2(x)$ are linearly independent if neither y_1 nor y_2

is a constant multiple of the other. For example

$$y_1(x) = x^2 \text{ and } y_2(x) = \sin(x).$$

Fact: If $y_1(x)$ and $y_2(x)$ are linearly independent solutions to the 2nd order homogeneous linear equation

$$P(x) \frac{d^2y}{dx^2} + Q(x) \frac{dy}{dx} + R(x)y = 0$$

then the general solution is given by

$$y(x) = c_1 y_1(x) + c_2 y_2(x)$$

where c_1 and c_2 are arbitrary constants.

From now on assume $P(x) = a$, $Q(x) = b$,
 $R(x) = c$, when a, b and c are constants. Thus
we wish to solve

$$a \frac{d^2y}{dx^2} + b \frac{dy}{dx} + cy = 0.$$

Also assume that $a \neq 0$.

Let's try $y = e^{rx}$ when r is constant.

Plugging this function into the left hand side gives

$$ar^2e^{rx} + bre^{rx} + ce^{rx} = (ar^2 + br + c)e^{rx}$$

Notice that if we choose r such that $ar^2 + br + c = 0$
then we get a solution. Fantastic! We call
this quadratic equation the characteristic equation
of the above differential equation. Thus solving
the differential equation is closely related to
finding the roots of $ar^2 + br + c = 0$. The
quadratic formula gives the two roots as:

$$r_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$$

$$r_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

Case 1 $b^2 - 4ac > 0$. In this case r_1 and r_2 are real numbers and $r_1 \neq r_2$. This gives linearly independent solutions $e^{r_1 x}$ and $e^{r_2 x}$ (they are linearly independent because $r_1 \neq r_2$), hence the general solution is $y(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x}$.

Example $\frac{dy}{dx} - 3 \frac{dy}{dx} + 2y = 0$. Need to find

roots of $r^2 - 3r + 2r = 0$. Note

$$r_1 = \frac{3 + \sqrt{9 - 8}}{2} = 2, \quad r_2 = \frac{3 - \sqrt{9 - 8}}{2} = 1$$

Thus a general solution is $y(x) = c_1 e^{2x} + c_2 e^x$,

where c_1 and c_2 are constants.

Case 2 $b^2 - 4ac = 0$. Here there is only one root

$r_1 = r_2 = \frac{-b}{2a}$. This only gives one solution $y = c_1 e^{r_1 x}$.

We need to find another. Let's try $y_2 = xe^{r_1 x}$.

$$\frac{d^2y_2}{dx^2} = 2r_1 e^{r_1 x} + r_1^2 x e^{r_1 x}$$

$$\frac{dy_2}{dx} = e^{r_1 x} + r_1 x e^{r_1 x}$$

$$\text{Thus } a \frac{d^2y_2}{dx^2} + b \frac{dy_2}{dx} + cy_2 = a(2r_1 e^{r_1 x} + r_1^2 x e^{r_1 x}) \\ + b(e^{r_1 x} + r_1 x e^{r_1 x}) \\ + cxe^{r_1 x} \\ r_1 = \frac{-b}{2a} \quad \begin{matrix} = (2ar_1 + b)e^{r_1 x} + (ar_1^2 + br_1 + c)x e^{r_1 x} \\ \uparrow \\ = 0 \cdot e^{r_1 x} + 0 \cdot x e^{r_1 x} \\ = 0 \end{matrix} \\ \text{remember}$$

Thus $y_2(x) = xe^{r_1 x}$ is another solution. Note also it is linearly independent from $y_1(x) = e^{r_1 x}$. Thus a general solution is $y(x) = c_1 e^{r_1 x} + c_2 x e^{r_1 x}$, where c_1 and c_2 are constants.

Example $\frac{dy}{dx^2} - 2 \frac{dy}{dx} + y = 0$. Need to find roots of

$$r^2 - 2r + 1 = 0 \Rightarrow r_1 = r_2 = 1$$

Thus a general solution is $y(x) = C_1 e^{\alpha x} + C_2 x e^{\alpha x}$

Case 3 $b^2 - 4ac < 0$. In this case there are

no real roots to $ar^2 + br + c = 0$, so it would seem at first that there is nothing we can do. Let's not give up just yet though. Let

$$\alpha = \frac{-b}{2a} \quad \text{and} \quad \beta = \frac{\sqrt{4ac - b^2}}{2a} \quad \begin{matrix} \text{makes sense} \\ \text{because} \end{matrix}$$
$$b^2 - 4ac < 0$$

Claim : $y_1(x) = e^{\alpha x} \sin \beta x$ and $y_2(x) = e^{\alpha x} \cos \beta x$

are solutions.

function

To see this just differentiate each ~~one~~ twice and feed them into the left hand side of the differential equation. e.g.

$$\del{y_1} \frac{dy_1}{dx} = \alpha e^{\alpha x} \sin \beta x + \beta e^{\alpha x} \cos \beta x$$

$$\frac{d^2 y_1}{dx^2} = \alpha^2 e^{\alpha x} \sin \beta x + \alpha \beta e^{\alpha x} \cos \beta x$$

$$+ \alpha \beta e^{\alpha x} \cos \beta x - \beta^2 e^{\alpha x} \sin \beta x$$

$$= (\alpha^2 - \beta^2) e^{\alpha x} \sin \beta x + 2\alpha \beta e^{\alpha x} \cos \beta x.$$

$$\begin{aligned}
 \Rightarrow & a \frac{dy_1}{dx^2} + b \frac{dy_1}{dx} + cy_1 \\
 = & a ((\alpha^2 - \beta^2) e^{\alpha x} \sin \beta x + 2\alpha\beta e^{\alpha x} \cos \beta x) \\
 & + b (\alpha e^{\alpha x} \sin \beta x + \beta e^{\alpha x} \cos \beta x) \\
 & + c (e^{\alpha x} \sin \beta x) \\
 = & (a(\alpha^2 - \beta^2) + b\alpha + c) e^{\alpha x} \sin \beta x \\
 & + (2\alpha\beta + b\beta) e^{\alpha x} \cos \beta x
 \end{aligned}$$

It is possible to show that

$$a(\alpha^2 - \beta^2) + b\alpha + c = 0 \quad \text{and} \quad 2\alpha\beta + b\beta = 0,$$

giving the result (just substitute $\alpha = \frac{-b}{2a}$ and $\beta = \sqrt{\frac{4ac - b^2}{4a}}$ into each equation). The same

basic calculation holds for $y_2(x) = e^{\alpha x} \cos \beta x$.

Thus a general solution is given by

$$y(x) = c_1 e^{\alpha x} \sin \beta x + c_2 e^{\alpha x} \cos \beta x.$$

Example: A spring's motion is modelled by the 2nd order differential equation:

$$m \frac{d^2x}{dt^2} = -\frac{k}{l} x \quad (m, k > 0)$$

mass &
weight Spring constat.

Let's solve this: $m \frac{d^2x}{dt^2} = -kx \Rightarrow m \frac{d^2x}{dt^2} + kx = 0$

Thus need to find roots of $mr^2 + k = 0$. No real roots so in case 3.

$$\alpha = \frac{-b}{2a} = 0 \quad \beta = \frac{\sqrt{4ac - b^2}}{2a} = \frac{\sqrt{4mk}}{2m}$$

$$= \frac{\sqrt{k}}{\sqrt{m}}$$

thus a general solution is

$$y(x) = c_1 \sin \sqrt{\frac{k}{m}} t + c_2 \cos \sqrt{\frac{k}{m}} t$$

This gives simple harmonic motion as expected.

The real reason that sin and cos show up in case 3 is because of the fact that the roots of $ar^2 + br + c = 0$ are not real numbers but

complex numbers. I decided not to take this approach here because it would be very confusing to those of you who haven't seen complex numbers before. If you are interested go and ~~read~~ read appendix H. There's nothing complex about them they are just a way of extending numbers from a line to a plane! If I have time I'll talk about them at the very end of the course.