

Lecture 33 : Predator-Prey Systems

Imagine we have two populations, represented by the functions R and W . Assume that the population given by W are predators and the population given by R are the prey. Also assume that in the absence of any predators, the prey population follows the natural growth law, that is :

$$\text{If } W(t) = 0 \Rightarrow \frac{dR}{dt} = kR \text{ where } k > 0$$

Now assume that in the absence of any prey, the predator population declines at a rate directly proportional to its size, that is,

$$\text{If } R(t) = 0 \Rightarrow \frac{dW}{dt} = -rW \text{ where } r > 0$$

We also assume that the two populations encounter each other at a rate directly proportional to both their sizes, and therefore directly proportional to RW .

A system of two differential equations that incorporates all of these assumptions is given by :

$$\frac{dR}{dt} = kR - aRW, \quad \frac{dW}{dt} = -rW + bRW,$$

where k, a, b, r are all positive constants.

These are known as the Predator-Prey equations.

A solution to this system is a pair of functions $R(t)$ and $W(t)$ that simultaneously satisfy both equations.

Warning: Strictly speaking the predator-prey equations are not differential equations in this sense of our ~~usual~~ usual definition because each involves 2 unknown functions instead of just 1.

What the predator-prey equations really are is a system of two differential equations in two unknowns.



How could we go about finding solutions to a predator-prey system? We should try and combine the two equations into a single differential

equation. How could we do this? We will use the chain rule to eliminate t as follows:

$$\frac{dW}{dt} = \frac{dW}{dR} \cdot \frac{dR}{dt}$$

$$\Rightarrow \frac{dW}{dR} = \frac{\left(\frac{dW}{dt}\right)}{\left(\frac{dR}{dt}\right)} = \frac{-rW + bRW}{kR - aRW}$$

Fantastic! We've now got a single differential equation that we can try and solve. Fortunately it is separable, so we could separate variables and solve as in §9.3. When we do this though the answer will be hard to visualise. You should try doing this yourself. (it's not too hard). Instead let's use direction fields. Let's do it by example.

Look at example 1 on page 623. Here $b = 0.08$, $a = 0.001$, $r = 0.02$ and $k = 0.00002$. Thus we need to draw the direction field for

$$\frac{-0.02W + 0.00002RW}{0.08R - 0.001RW}$$

in the RW -plane. This is done in Figure 1 on page 624. A few remarks are needed here.

$$\text{If } W(t) = 80 \Rightarrow 0.08 - 0.001W = 0$$

$$\Rightarrow \frac{dR}{dt} = 0.08R - 0.001RW = 0$$

$$\text{If } R(t) = 1000 \Rightarrow -0.02 + 0.00002R = 0$$

$$\Rightarrow \frac{dW}{dt} = -0.02W + 0.00002RW = 0$$

Thus because $\frac{dW}{dR} = \left(\frac{dW}{dt} \right) / \left(\frac{dR}{dt} \right)$ we draw vertical line segments if $W(t) = 80$ and $R(t) \neq 1000$.

At the point $W = 80$ and $R = 1000$ there is no sensible value for $\frac{dW}{dR}$ and we just have a dot.

Now given an initial R population R_0 , and an initial W population W_0 we trace the parallel path to the direction field passing through (R_0, W_0) .

Various examples are given by Figure 2 on page 624.

We call the RW -plane the phase plane

and solution curves phase trajectories. The phase trajectory is the path traced out as time passes.

We see that for ~~in~~ general starting populations the populations will oscillate back and forth out of sync with each other. Note also at the point $R=1000$, $W=80$ both populations are in perfect equilibrium.

A downside to this method is that it fails to tell us what the populations will be at a given time.

