

Lecture 31 : Models for Population Growth

Imagine we have a population and we would like to understand how it changes over time. Perhaps we would like to know if a population reaches some kind of equilibrium over time. Perhaps we know what a population is at a given time and would like to know the starting population. To answer such questions we must first model the growth of the population and then attempt to understand the behavior of the model. As in the first lecture on differential equations fix the following notation:

$t$  = time

$P(t)$  = population size at time  $t$ .

The Law of Natural Growth

Model : The rate of growth of the population is directly proportional to the size of the population.

Mathematical Interpretation :  $\frac{dP}{dt} = kP$ ,

where  $k > 0$  is some constant.

Solution : This is a separable differential equation so we solve as follows :

$$\int \frac{dp}{p} = \int k dt$$

$$\Rightarrow \ln |p| = kt + C$$

$$\Rightarrow p = \pm e^C \cdot e^{kt}$$

Hence a general solution is given by  $P(t) = Ae^{kt}$ , where  $A$  is any constant.

Initial Conditions : If  $P(0) = P_0$ , for some positive

number  $P_0$ , we can restrict to a unique solution as follows:

$$P_0 = P(0) = A e^{k \cdot 0} = A.$$

Hence  $P(t) = P_0 e^{kt}$ .

We can see that if  $P_0 > 0$  then  $P(t) \rightarrow \infty$  as  $t \rightarrow \infty$ .

Example : Assume a population grows subject to the

differential equation  $\frac{dp}{dt} = 0.2p$  and at  $t = 10$ ,

$P(10) = 100$ . What is  $P_0$ ?

We know  $P(t) = P_0 e^{0.2t}$  and  $P(10) = P_0 e^{2} = 100$

$$\Rightarrow P_0 = \frac{100}{e^2} \approx 14$$

## The Logistic Model

Model : For small  $P(t)$  the growth rate follows the Law of natural growth, i.e.  $\frac{dP}{dt} \approx kP$  for some constant  $k > 0$ . However if  $P(t) > M$  the population size declines, i.e.  $\frac{dP}{dt} < 0$ .

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right)$$

Important Remark : We've chosen  $kP\left(1 - \frac{P}{M}\right)$  because it is the simplest function in  $P$  which does what the model describes. There are many other possibles, e.g.  $kP\left(1 - \frac{P}{M}\right)^3$ .

Solution Again the differential equation is separable, hence we solve it as follows :

$$\int \frac{dP}{P\left(1 - \frac{P}{M}\right)} = \int k dt \Rightarrow$$

$$\int \frac{M}{P(M-P)} dP = \int k dt \Rightarrow$$

$$\int \left(\frac{1}{P} + \frac{1}{M-P}\right) dP = \int k dt \Rightarrow$$

$$\ln|P| - \ln|M-P| = kt + C \Rightarrow$$

$$\ln \left| \frac{P}{M-P} \right| = kt + C$$

$$\Rightarrow \ln \left| \frac{M-P}{P} \right| = -kt - C$$

$$\Rightarrow \left| \frac{M-P}{P} \right| = e^{-C} \cdot e^{-kt}$$

$$\Rightarrow \frac{M-P}{P} = \pm e^{-C} \cdot e^{-kt}$$

$$\Rightarrow \frac{M-P}{P} = A \cdot e^{-kt} \quad (A = \pm e^{-C})$$

$$\Rightarrow P(t) = \frac{M}{1 + A e^{-kt}}$$

This gives a general solution.

Initial Conditions Assume  $P(0) = P_0 \Rightarrow$

$$P_0 = \frac{M}{1+A} \Rightarrow A = \frac{M-P_0}{P_0}$$

Observe that  ~~$M-P$~~   $e^{-kt} \rightarrow 0$  as  $t \rightarrow \infty$ .

Hence given any starting population size  $P_0$ ,  $P(t) \rightarrow M$

as  $t \rightarrow \infty$ . If we plot the direction field for  $kP(1-\frac{P}{M})$  we can also see this. For a specific

example look at Figure 2 on page 608.

## Other Models

Perhaps population growth is dependent on external factors such as the time of year. We could further enhance the logistic model to take this into account

as follows :

$$\frac{dp}{dt} = kp\left(1 - \frac{p}{M}\right) + \frac{\sin(t)}{P}$$

represents how changing seasons might affect population growth.

Perhaps a ~~pop~~ population begins to decline if it gets too small, perhaps below  $m$ . Then we could model this as follows :

$$\frac{dp}{dt} = kp\left(1 - \frac{p}{M}\right)\left(1 - \frac{m}{p}\right)$$

$\uparrow$   
becomes negative if  $p(t) < m$ .

In general, there are many ways to model population growth. <sup>The</sup> more factors we include in our model, consequently the more complicated the differential equation, and / more difficult it will be to find an explicit solution.