

Lecture 30 : Separable Equations

Last time we studied solutions to differential equations of the form

$$y' = F(x, y).$$

Today we will specialise further and assume

$$F(x, y) = \frac{g(x)}{h(y)},$$

where  $g$  and  $h$  are two functions.

Examples  $y' = \frac{x}{y}$ ,  $y' = \frac{\sin(x)}{\cos(y)}$

Let us switch to Leibnitz notation :

$$\frac{dy}{dx} = \frac{g(x)}{h(y)}$$

Let's further rearrange this into differential form:

$$h(y) dy = g(x) dx$$

Don't worry too much what this means, the main thing is we can integrate:

$$\int h(y) dy = \int g(x) dx$$

Let's really check this is true:

$$\begin{aligned} \frac{d}{dx} \int h(y) dy &= \underbrace{\frac{d(\int h(y) dy)}{dy}}_{\text{Chain Rule}} \cdot \frac{dy}{dx} \\ &= h(y) \cdot \frac{dy}{dx} \\ \frac{d}{dx} \int g(x) dx &= g(x) \end{aligned}$$

Thus  $\int h(y) dy = \int g(x) dx \Rightarrow h(y) \cdot \frac{dy}{dx} = g(x)$

$$\Rightarrow \frac{dy}{dx} = \frac{g(x)}{h(y)}$$

Thus we have a nice method to solve differential equations of this form.

Example  $\frac{dy}{dx} = \frac{x^2}{y^2} \Rightarrow$  we should solve

$$\int y^2 dy = \int x^2 dx \Rightarrow \frac{1}{3} y^3 = \frac{1}{3} x^3 + C$$

constant.

Hence the general solution is

$$y = \sqrt[3]{x^3 + K},$$

where  $K = 3C$ , is any constant.

If we impose the initial condition  $y(0) = 2$

$$\Rightarrow 2 = \sqrt[3]{K} \Rightarrow K = 8, \text{ thus}$$

$$y = \sqrt[3]{x^3 + 8} \quad \text{solves} \quad \frac{dy}{dx} = \frac{x^2}{y^2} \quad \text{with} \\ y(0) = 2.$$

Let's try solving  $\frac{dy}{dx} = x^2y = \frac{x^2}{(\frac{1}{y})}$ .

Thus we should solve  $\int \frac{1}{y} dy = \int x^2 dx$

$$\Rightarrow \ln|y| = \frac{1}{3}x^3 + C \Rightarrow |y| = e^C \cdot e^{\frac{1}{3}x^3}$$

$\Rightarrow y = \pm A e^{\frac{1}{3}x^3}$ , where  $A = e^C$  is an arbitrary ~~positive~~ positive constant. Note that we can remove the  $\pm$ , by assuming  $A$  is any constant.

Thus a general solution is  $y = A e^{\frac{1}{3}x^3}$

Depending on  $A$  we will get different graphs. We can sketch them as in Example 3 on page 596.

### Orthogonal Trajectories

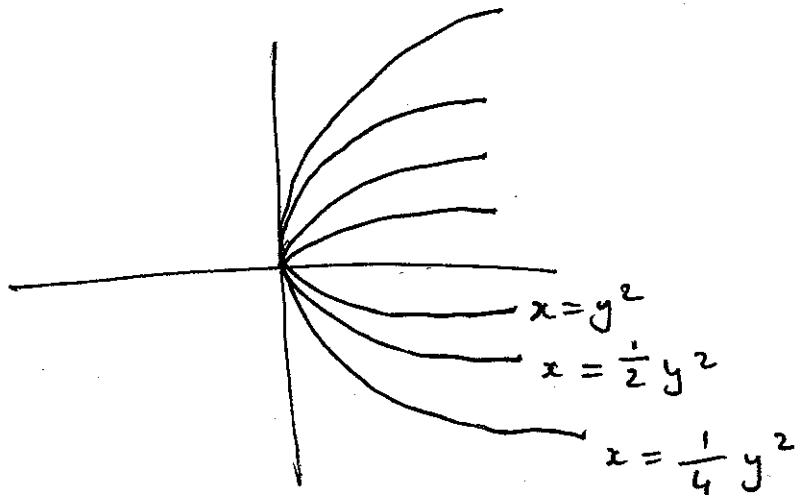
to a family of curves

An orthogonal trajectory is a curve that intersects each curve of the family at right angles (also called orthogonally). See Figures 7 and 8 on page 597.

Given a family of curves how can we construct an orthogonal trajectory?

First observe that if  $y = mx + c$  then any line at right angles to it has gradient/slope  $-\frac{1}{m}$ . We shall use this property.

Example: Find ~~the~~ an orthogonal trajectory to the family of curves  $x = ky^2$ , where  $k$  is an arbitrary constant.



If our curve is to cross each curve at right angles it must be orthogonal to the tangent line of any curve in the family at an intersection.

Thus for the curve  $x = ky^2$  we must determine

$\frac{dy}{dx}$ . By implicit differentiation we know

$$1 = \frac{dx}{dx} = \frac{d(ky^2)}{dx} = \frac{dky^2}{dy} \cdot \frac{dy}{dx} \\ = 2ky \cdot \frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2ky}$$

We would like to remove the unknown  $k$  from the equation. Observe that  $x = ky^2 \Rightarrow k = \frac{x}{y^2}$

From every curve in the family satisfies

$$\frac{dy}{dx} = \frac{1}{2\left(\frac{x}{y^2}\right)y} = \frac{y}{2x}.$$

Thus an orthogonal trajectory must satisfy

$$\frac{dy}{dx} = -\frac{2x}{y}$$

We can solve this!

$$\int y dy = -2 \int x dx \Rightarrow \frac{1}{2} y^2 = -\frac{2}{2} x^2 + C \\ \Rightarrow \frac{y^2}{2} + x^2 = C$$

For any constant  $C$  is an orthogonal trajectory!

See Figure 9 on page 598 for what these look like.