

# Math 1B : Calculus (Fall 2014)

## Lecture 29 : Direction Fields and Euler's Method

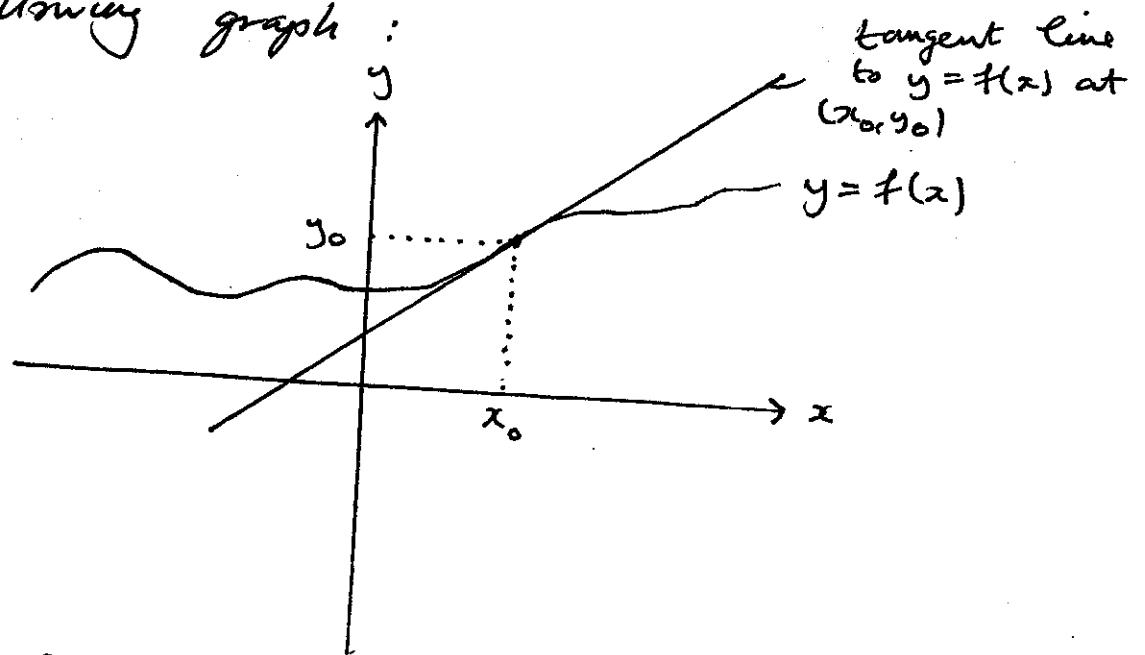
Today we will consider how to visualise solutions to differential equations of the form

$$y' = F(x, y),$$

where  $F(x, y)$  is some expression in  $x$  and  $y$ .

In general it is too hard to find an explicit formula for the solution. We can however learn a lot about the solution through a graphical approach.

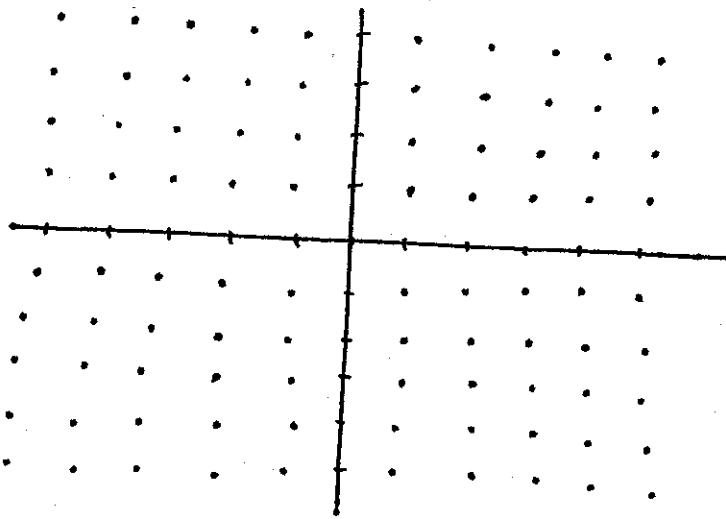
Imagine that we had a solution  $f(x)$  and it had the following graph :



Because  $f(x)$  is a solution to  $y' = F(x, y)$  we know

that the gradient of the tangent line at  $(x_0, y_0)$  must be equal to  $F(x_0, y_0)$ . This suggests the following visual interpretation of the solution:

Step 1 Mark out a grid on the  $x$ - $y$  plane, e.g.



Step 2. For each point in the grid draw a small line segment of gradient  $F(x_0, y_0)$ , where  $(x_0, y_0)$  is the point of the grid, e.g. If

$$F(x, y) = x^2 + y^2 - 1 \text{ see figure 7 on page 587 of the textbook.}$$

This is called a direction field. I like to

think about it as a flow of water on a surface, where the line segments indicate the

direction of flow.

Step 3 If the initial conditions are  $F(x_0) = y_0$ , draw a curve parallel to the direction of flow going through the point  $(x_0, y_0)$ . For example if  $F(x, y) = x^2 + y^2 - 1$  and  $x_0 = y_0 = 0$  then Figure 6 on page 587 shows the appropriate curve. I like to think about this as dropping ink in the water at  $(x_0, y_0)$  and observing the path it traces out.

This graph is the graph of the solution to  $y' = F(x, y)$  with initial conditions  $F(x_0) = y_0$ .

This method is useful when we want to get some idea of the general behavior of a solution, e.g. is it bounded or not.

We can expand upon this idea to actually numerically approximate  $f(x)$ . This is called Euler's Method.

Step 1 Fix a number  $h > 0$ , called a step size

Step 2 Starting at  $(x_0, y_0)$  draw a straight line with gradient  $F(x_0, y_0)$  with horizontal length  $h$ .

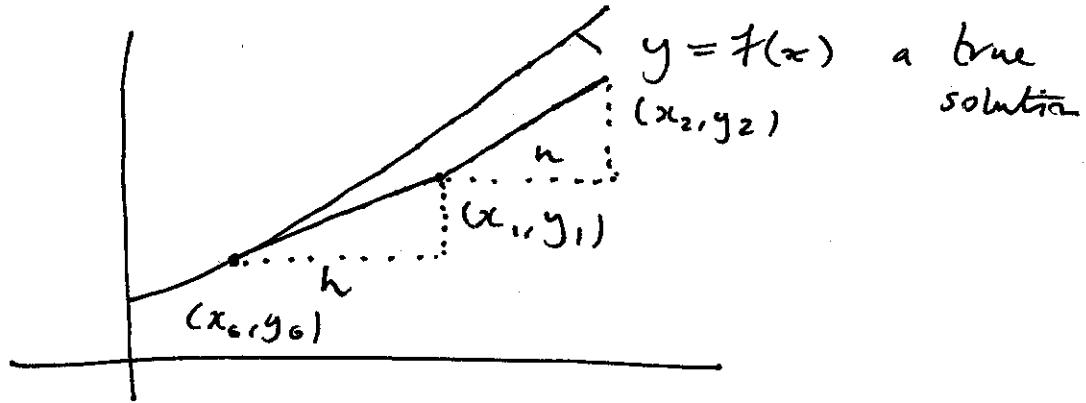
Step 3 At the endpoint of this line  $(x_1, y_1)$  draw a new straight line with gradient  $F(x_1, y_1)$  and horizontal length  $h$ .

Step 4 Continue this process. In general:

$$x_n = x_{n-1} + h \text{ and}$$

$$y_n = y_{n-1} + h F(x_{n-1}, y_{n-1})$$

Picture :



Clearly, for smaller step size this will approximate the true solution more accurately.

For a detailed example of how to use this method look at example 3 on page 590.