Math 1B : Calculus (Fall 2014)

Lecture 28 : Modelling with Differential Equations

A differential equation is an equation that contains an unknown function and one or more of its derivatives.

Example: If we represent an unknown function in the variable x by y then the following are all differential equations :

$$y' = y \; ;$$

$$y'' = y' + y \; ;$$

$$\frac{dy}{dx} = xy + 2 \; ;$$

$$\left(\frac{d^2y}{dx^2}\right)^3 + x\left(\frac{dy}{dx}\right)^4 = 1 .$$

The <u>order</u> of a differential equation is the order of the highest derivative that occurs in the equation. In the above example $y' = y$ has order 1, whereas $\left(\frac{d^2y}{dx^2}\right)^3 + x\left(\frac{dy}{dx}\right)^4 = 1$ has order 2.

A function $f(x)$ is called a <u>solution</u> of a differential equation if the equation is satisfied when $y = f(x)$ and its derivatives are substituted into the equation.

<u>Example</u> : Take the differential equation $y' = y$.
Clearly $y = e^x$ is a solution. Also note though that for any constant number C, $y = Ce^x$ is also a solution.

This example shows that in general a differential equation does <u>not</u> have a unique solution (if indeed it has a solution).
We shall see later that in order to get a unique solution we shall need to fix some initial <u>conditions</u>.
In the above example this means choosing some specific value for $y(0)$. For example if we want a solution such that $y(0) = 1$, then $y = e^x$ will turn out to be the <u>unique</u> solution.

To summarise :

A given differential equation, in general, will have not one, but a <u>family</u> of solutions. However, after imposing certain <u>initial conditions</u> on the solutions (i.e. values of $f(x)$ and higher derivatives at a given x_0) there will be a unique solution.

To see a more complicated example than the one I've just given look at example 1 and 2 on pages 583 and 584.

<u>Very important Question</u>: Why do we care about differential equations?

The basic reason is because many natural systems are governed by differential equations.

<u>Population Growth</u>

A simple model of population growth is based on the assumption that the population grows at a rate proportional to the size of a population. Let's try and model this:

t = time (the independent variable)

$P(t)$ = population at time t.

The rate of change of the population is given by $\frac{dP}{dt}$.

Our model for population growth now becomes the differential equation

$$\frac{dP}{dt} = \underset{\substack{\uparrow \\ \text{proportionality} \\ \text{constant.}}}{k} P$$

This is just a differential equation! If we try and solve it we get the following family of solutions

$$P(t) = Ce^{kt},$$

where C is any constant. Imagine we now impose the initial condition $P(0) = 100$. Then

$$100 = P(0) = Ce^{k \cdot 0} = C$$

Hence the only solution to this differential equation is

$$P(t) = 100e^{kt}.$$

Fantastic! By completely solving this differential equation

We can determine the population in our model at <u>any</u> given time, for <u>any</u> starting population.

Of course this model is very very simplistic. What about if past a certain population, lets say M, the population begins decreasing due to constraints in the environment, i.e.

1) $\frac{dP}{dt} \approx kP$ if P is small

2) $\frac{dP}{dt} < 0$ if $P > M$

In the 1840s a famous Dutch mathematical biologist proposed the following differential equation to model this:

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right)$$

Finding the family of solution to this would give us a detailed understanding of this more sophisticated population model. We shall do this in §9.4.

This is just one example of how differential equations model physical systems. Throughout §9 we will encounter many more.

<u>Warning</u>: In the examples I have given it has been fairly easy to spot a family of solutions. In general it is <u>tremendously</u> difficult to solve a differential equation. Almost all of §9 and §17 will be devoted to doing this in very specific situations.