Math 1B: Calculus (Fall 2014)

Lecture 24: Taylor and Maclaurin Series (Part 1)

Under what circumstances can we represent a function $f(x)$ by a power series? If we are given a function $f(x)$ can we construct a power series that represents it? Last time we saw that manipulating

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots, \quad |x| < 1$$

gave us an answer in various special cases. Let's try and find a general answer.

Assume $f(x)$ can be represented by a power series as follows:

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \dots$$

for $|x-a| < R$. Can we determine the coefficients

(R: radius of convergence)

c_n, purely in terms of $f(x)$?

First observe that $f(a) = c_0 + c_1(a-a) + c_2(a-a)^2 + \dots$

$= c_0$

That was easy. How about higher coefficients?

$$f'(x) = \sum_{n=0}^{\infty} c_n n (x-a)^{n-1} \quad (\text{for } |x-a| < R)$$

$$= c_1 + c_2 2(x-a) + c_3 3 (x-a)^2 + \dots$$

Hence

$$f'(a) = c_1 + c_2 2(a-a) + c_3 3(a-a)^2 + \dots$$

$$= c_1$$

Lets try again!

$$f''(x) = \sum_{n=0}^{\infty} c_n n(n-1)(x-a)^{n-2}$$

$$= 2! c_2 + c_3 \times 3 \times 2 \times (x-a) + c_4 \times 4 \times 3 \times (x-a)^2 + \dots$$

Hence $f''(a) = 2! c_2 \Rightarrow c_2 = \dfrac{f''(a)}{2!}$

If we go again we'll find

$$f'''(a) = 3! c_3 \Rightarrow c_3 = \frac{f'''(a)}{3!}$$

Awesome! For general n we get

$$c_n = \frac{f^{(n)}(a)}{n!}$$ ← f differentiated n times and evaluated at a.

Conclusion: If $f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$

for $|x-a| < R \Rightarrow c_n = \frac{f^{(n)}(a)}{n!}$ for $n \geq 0$.

i.e. $f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots$

Definition Given a function $f(x)$, the Taylor series of $f(x)$ at a (a number) is the power series

$$f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots$$

In the special case $a = 0$ the power series is

$$f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \dots$$

We call this the Maclaurin series of $f(x)$.

Warning: We've shown that if $f(x)$ can be represented by a power series about a, then the power series is just the Taylor series. However, there exist functions that are not represented by their Taylor series. This is a bit upsetting.

We'll try and understand which functions are represented by their Taylor series.

Examples 1) $f(x) = e^x \Rightarrow f^{(n)}(x) = e^x$

Hence Maclaurin series of e^x is the power series

$$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

2) $f(x) = \sin(x) \Rightarrow f'(x) = \cos(x) \Rightarrow f''(x) = -\sin(x)$

$\Rightarrow f'''(x) = -\cos(x) \Rightarrow f^{(4)}(x) = \sin(x)$.

Hence Maclaurin series for $\sin(x)$ is

$$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

3) $f(x) = \cos(x)$. We get a similar pattern to $\sin(x)$, but with even powers of x, ie.

$$1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$$

In the above examples we've only determined the Maclaurin series. We do not know yet whether they truly represent the functions they come from.

Recall what a Taylor series really means.

$$f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots$$

$$= \lim_{n \to \infty} \{T_n(x)\}, \quad \text{where}$$

$$T_n(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

$T_n(x)$ is just the sum of the first $n+1$ terms of the Taylor series. For example, if $f(x) = e^x$

$$T_1(x) = 1 + x, \quad T_2(x) = 1 + x + x^2, \quad T_3(x) = 1 + x + x^2 + x^3.$$

So we see that $f(x)$ would be represented by its Taylor series if

$$f(x) = \lim_{n \to \infty} \{T_n(x)\}.$$

Lets carefully think what this means. If

$$R_n(x) = f(x) - T_n(x) \quad \text{then}$$

$$\lim_{n \to \infty} \{T_n(x)\} = f(x) \implies \lim_{n \to \infty} \{R_n(x)\} = 0$$

Hence if we want to show that

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots$$

for some fixed x, we must show that

$$\lim_{n \to \infty} \{R_n(x)\} = 0$$

In the next lecture we'll see how to do this for certain $f(x)$.