

Lecture 19 : Alternating Series.

An alternating series is a series whose terms alternate between positive and negative.

Examples :

$$\sum_{n=1}^{\infty} (-1)^{n-1} \cdot \frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{5} + \dots$$

$$\sum_{n=1}^{\infty} \frac{\sin\left(\frac{(2n+1)\pi}{2}\right)}{1+n^2} = -\frac{1}{2} + \frac{1}{5} - \frac{1}{10} + \dots$$

Interesting question : When does an alternating series converge?

Alternating Series Test

Let  $\{a_n\}$  be a sequence of positive numbers such that

- 1)  $a_{n+1} \leq a_n$  for all  $n \geq 1$
- 2)  $\lim_{n \rightarrow \infty} \{a_n\} = 0$

Then the alternating series  $\sum_{n=1}^{\infty} (-1)^{n-1} a_n = a_1 - a_2 + a_3 - a_4 + \dots$  is convergent.

## Remarks

i) Warning: This will only work if the series is alternating.

Recall that if  $\{b_n\}$  is a sequence such that

$\sum_{n=1}^{\infty} b_n$  is convergent, then this implies  $\lim_{n \rightarrow \infty} \{b_n\} = 0$ .

However, the converse is not true in general, meaning that just because  $\lim_{n \rightarrow \infty} \{b_n\} = 0$ , it does not follow that  $\sum_{n=1}^{\infty} b_n$  is convergent. The standard example is

the harmonic series  $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots$ .

2) As with the integral test and the comparison tests, the alternating series test only tells us that a series converges or diverges. It does not tell us what the value is if convergent.

## Examples

$$1) 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} \dots = \sum_{n=1}^{\infty} (-1)^{n-1} \cdot \frac{1}{n}$$

convergent by alternating series test, because

$$\text{i)} \frac{1}{n+1} < \frac{1}{n} \text{ for all } n \geq 1$$

$$\text{ii)} \lim_{n \rightarrow \infty} \left\{ \frac{1}{n} \right\} = 0.$$

$$2) \sum_{n=1}^{\infty} (-1)^{n-1} \frac{n^2}{n^3 + 1}$$

Observe that this is an alternating series. Let's try and use the alternating series test. Let

$a_n = \frac{n^2}{n^3 + 1}$  for all  $n \geq 1$ . We first need to show

$\{a_n\}$  is decreasing. Let's use calculus! The

sequence obviously comes from the function  $f(x) = \frac{x^2}{x^3 + 1}$ .

$f'(x) = \frac{x(2-x^3)}{(x^3+1)^2}$ . Note that for  $x > \sqrt[3]{2}$ ,

$f'(x) < 0$ , hence  $f(x)$  is decreasing on  $(\sqrt[3]{2}, \infty)$ .

Thus  $a_2 > a_3 > a_4 > a_5 \dots$

We could also verify that  $a_1 > a_2$ , but all that really matters is that  $\{a_n\}$  is eventually decreasing.

Now observe that

limit of  
sequence  
↓

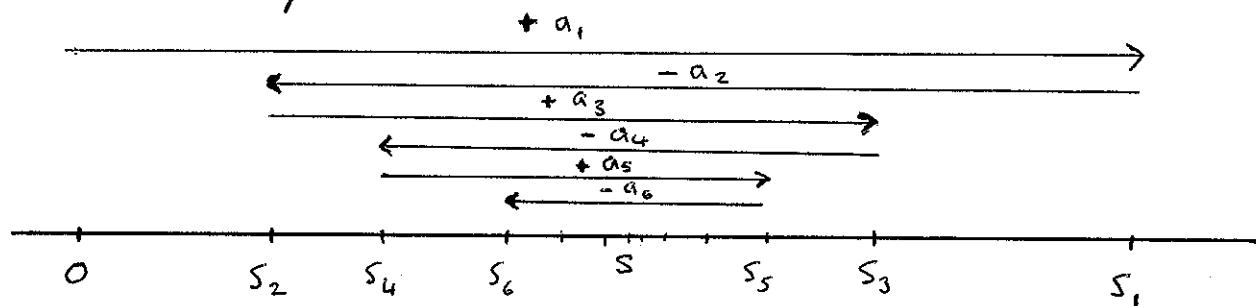
limit of  
function  
↓

$$\lim_{n \rightarrow \infty} \{a_n\} = \lim_{n \rightarrow \infty} \left\{ \frac{n^2}{n^3 + 1} \right\} = \lim_{x \rightarrow \infty} \frac{x^2}{x^3 + 1} = 0$$

Hence convergent by alternating series test. L'Hopital's done twice.

Why is the alternating series test true?

Let's draw a picture:



$$S_1 = a_1$$

$$S_2 = a_1 - a_2$$

$$S_3 = a_1 - a_2 + a_3$$

$$S_4 = a_1 - a_2 + a_3 - a_4$$

:

Consider the sequence of even partial sums:

$$\{S_2, S_4, S_6, S_8, \dots\}$$

Note that this sequence is increasing and bounded above by  $S_1 = a_1$ . Hence by the monotone sequence theorem (§II.1) it is convergent with limit which we denote by  $S$ .

Recall that  $\sum_{n=1}^{\infty} (-1)^{n-1} a_n$  is the limit of the whole

sequence  $\{S_1, S_2, S_3, S_4, \dots\}$ . We need to think about the odd terms also.

Notice that

$$S_{2n+1} = S_{2n} + a_{2n+1}, \text{ hence}$$

$$\begin{aligned}\lim_{n \rightarrow \infty} \{S_{2n+1}\} &= \lim_{n \rightarrow \infty} \{S_{2n} + a_{2n+1}\} \\ &= \lim_{n \rightarrow \infty} \{S_{2n}\} + \lim_{n \rightarrow \infty} \{a_{2n+1}\} \\ &= s + 0 \\ &= s.\end{aligned}$$

Since both even and odd partial sums converge to  $s$ , we deduce that  $\lim_{n \rightarrow \infty} \{s_n\} = s$  and that

$$\sum_{n=1}^{\infty} (-1)^{n-1} a_n \text{ converges.}$$

A useful thing we can see from this is that

if  $s = \sum_{n=1}^{\infty} (-1)^{n-1} a_n$ , where

i)  $a_{n+1} \leq a_n \quad (a_n \geq 0)$

ii)  $\lim_{n \rightarrow \infty} \{a_n\} = 0$

then  $|s - s_n| \leq a_{n+1}$

We can use this in the following way:

Estimate the sum  $\sum_{n=1}^{\infty} (-1)^{n-1} \cdot \frac{1}{n^6}$  with an error at less than  $\frac{1}{1000}$ .

First note that the series is convergent by the alternating series test.

$$S_1 = 1, \quad |S - S_1| \leq a_2 = 0.015625$$

$$S_2 = 1 - \frac{1}{2^6}, \quad |S - S_2| \leq a_3 = 0.0013717$$

$$S_3 = 1 - \frac{1}{2^6} + \frac{1}{3^6}, \quad |S - S_3| \leq a_4 = 0.000244 < \frac{1}{1000}$$

$$\text{Hence } S_3 = 1 - \frac{1}{2^6} + \frac{1}{3^6} \approx 0.9857$$

$$\text{Hence } S \approx 0.9857 \text{ to within } \frac{1}{1000}$$

Warning This only works because the series is alternating. It would not work for general sequences.