

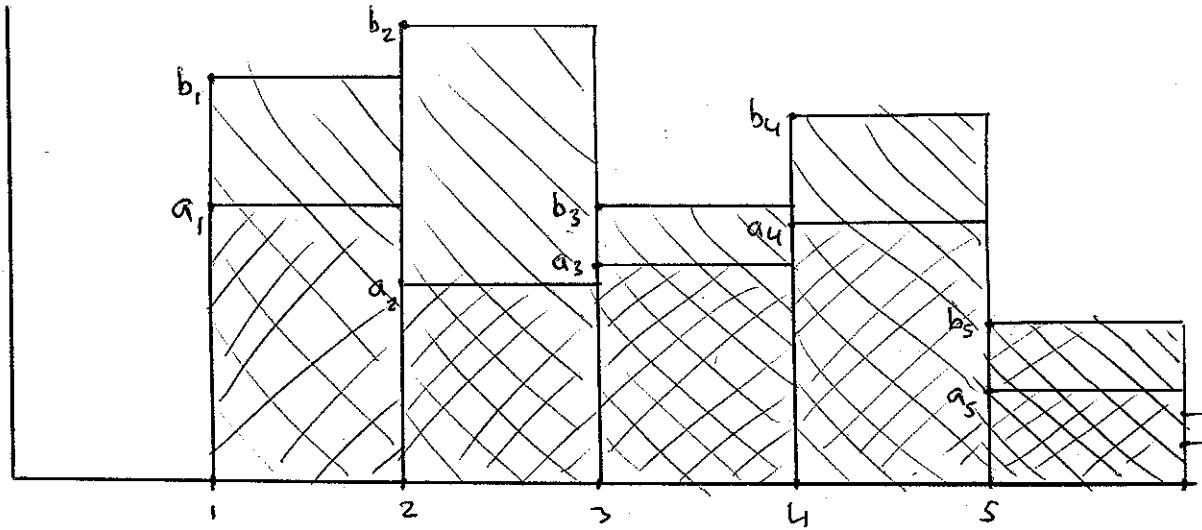
Math 1B : Calculus (Fall 2014)

Lecture 18 : The comparison tests

The comparison tests for series are very similar to the comparison tests for improper integrals.

Let $\{a_n\}$ and $\{b_n\}$ be 2 sequences , all of whose terms are positive . Assume that $a_n \leq b_n$ for all $n \geq 1$. Recall that we can visualize the infinite series

$$\sum_{n=1}^{\infty} a_n \text{ and } \sum_{n=1}^{\infty} b_n \text{ as areas in the following way :}$$



$$\text{area} = \sum_{n=1}^{\infty} a_n$$

$$\text{area} = \sum_{n=1}^{\infty} b_n$$

Note this is very similar to the picture we draw for improper integrals $\int_1^\infty f(x) dx$ and $\int_1^\infty g(x) dx$, when $f(x) \leq g(x)$.

The same logic as for improper integrals gives the comparison test for series:

$$A) \sum_{n=1}^{\infty} b_n \text{ convergent} \Rightarrow \sum_{n=1}^{\infty} a_n \text{ convergent}$$

$$B) \sum_{n=1}^{\infty} a_n \text{ divergent} \Rightarrow \sum_{n=1}^{\infty} b_n \text{ divergent.}$$

To use this we must find some convergent and divergent series. Our two main examples are

$$1) \sum_{n=1}^{\infty} \frac{1}{n^p} = \begin{cases} \text{divergent} & 0 \leq p \leq 1 \\ \text{convergent} & p > 1 \end{cases} \quad (\text{shown using integral test})$$

$$2) \sum_{n=1}^{\infty} ar^{n-1} = \begin{cases} \text{divergent} & |r| \geq 1 \\ \text{convergent} & |r| < 1 \end{cases} \quad (a \neq 0)$$

Example of use: Determine whether $\sum_{n=1}^{\infty} \frac{n+1}{n^2-2}$

is divergent or convergent.

Observe that $\frac{n+1}{n^2-2}$ is a linear over a quadratic so

we guess it behaves like $\sum \frac{1}{n}$ as a series. Thus we guess it should be divergent. Thus we want to bound it below by a divergent series. Note that

$$\frac{n+1}{n^2-2} > \frac{n+1}{n^2} > \frac{n}{n^2} = \frac{1}{n} \quad \text{for } n \geq 2.$$

Hence $\sum_{n=2}^{\infty} \frac{n+1}{n^2-2}$ is divergent by comparison test with

$$\sum_{n=2}^{\infty} \frac{1}{n}. \quad \text{Hence } \sum_{n=1}^{\infty} \frac{n+1}{n^2-2}.$$

This example illustrates an important ~~fact~~ fact:

In the comparison test we do not require $a_n \leq b_n$ for all $n \geq 1$. We only need to

know that there exist $N > 0$, such that $a_n \leq b_n$ for all $n \geq N$. In the above

example we had $N = 2$.

This is because $\sum_{n=1}^{\infty} a_n$ converges if and only if $\sum_{n=N}^{\infty} a_n$ converges. Exactly the same statement holds for divergence.

There are occasions where we want to use the comparison test, but it is not clear how to do so.

Example

$$\sum_{n=1}^{\infty} \frac{n^2 + 2n + 1}{n^4 - n + 1}$$

When we look at this we see quadratic over degree 4, so we guess it must behave like $\sum \frac{1}{n^2}$, which is convergent. So to use the comparison test we want to bound it above by a convergent series.

The obvious choice is $\frac{1}{n^2}$. However

$$\frac{1}{n^2} < \frac{n^2 + 2n + 1}{n^2} \leq \frac{n^2 + 2n + 1}{n^4 - n + 1}$$

for all $n \geq 1$. The inequalities are backwards so we can't deduce anything.

In this case we will use the following test:

The Limit Comparison Test

Suppose $\{a_n\}$ and $\{b_n\}$ are positive sequences. If

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = c,$$

where c is a finite number and $c > 0$, then either both $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ both diverge, or both

$$\sum_{n=1}^{\infty} a_n \text{ and } \sum_{n=1}^{\infty} b_n \text{ converge.}$$

Before giving a demonstration of why this is true,

let's use it on $\sum_{n=1}^{\infty} \frac{n^2 + 2n + 1}{n^4 - n + 1}$. Let $a_n = \frac{1}{n^2}$,

$$b_n = \frac{n^2 + 2n + 1}{n^4 - n + 1} \Rightarrow \frac{a_n}{b_n} = \frac{n^4 - n + 1}{n^4 + 2n^3 + n^2}$$

$$\lim_{n \rightarrow \infty} \left\{ \frac{n^4 - n + 1}{n^4 + 2n^3 + n^2} \right\} = \lim_{x \rightarrow \infty} \frac{x^4 - x + 1}{x^4 + 2x^3 + x^2} = 1 > 0.$$

\uparrow limit of sequence \uparrow limit at function use L'Hopital 4 times

We know $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges. So by limit comparison

test $\sum_{n=1}^{\infty} \frac{n^2 + 2n + 1}{n^4 - n + 1}$ convergent! Awesome.

Here's why it works:

Because $c > 0$, we can find m, M two positive numbers, such that $0 < m < c < M$. By definition of limits we know there is a number N , such that

$$m < \frac{a_n}{b_n} < M \quad \text{for all } n > N.$$

This means $m b_n < a_n < M b_n$ for all $n > N$.

$$\text{If } \sum_{n=1}^{\infty} b_n \text{ diverges} \Rightarrow \sum_{n=1}^{\infty} m b_n \text{ diverges}$$

$$\Rightarrow \sum_{n=1}^{\infty} a_n \text{ diverges}$$

↑

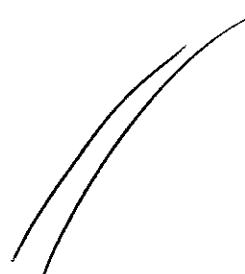
usual
comparison test.

$$\text{If } \sum_{n=1}^{\infty} b_n \text{ converges} \Rightarrow \sum_{n=1}^{\infty} M b_n \text{ converges}$$

$$\Rightarrow \sum_{n=1}^{\infty} a_n \text{ converges}$$

↑

usual
comparison test



Neat! This is quite hard to understand the first time you see it, because the first step involves properly understanding the definition of limit. I really recommend trying to though, because it is a really great piece of mathematics!