Let \( \{a_n\}_{n=1}^{\infty} \) be a sequence. Does it make sense to talk about the sum

\[
\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + a_4 + \ldots 
\]

Example 1) Let \( \{a_n\}_{n=1}^{\infty} \) be the sequence with \( a_n = \frac{1}{2^n} \), i.e., \( \{\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots\} \)

Let \( S_n = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots + \frac{1}{2^n} \)

What happens as \( n \to \infty \)?

From the picture we can guess the following:

\[
\lim_{n \to \infty} S_n = 1
\]

So even though it doesn't make sense to literally add together infinitely many terms, in this case it is sensible to write

\[
\sum_{n=1}^{\infty} a_n = 1
\]
2) Let \( \{a_n\}_{n=1}^{\infty} \) be the sequence \( a_n = n \), i.e., \( \{1, 2, 3, 4, \ldots\} \). If we do the same thing here then

\[
S_n = 1 + 2 + 3 + \ldots + n.
\]

Clearly \( S_n \geq n \) for all \( n \geq 1 \), hence the limit \( \lim_{n \to \infty} S_n \) does not have any meaning as a number in this case.

These observations inspire the following definition:

**Definition**

Let \( \{a_n\}_{n=1}^{\infty} \) be a sequence. We define the **\( n \)th partial sum** to be

\[
S_n = \sum_{i=1}^{n} a_i = a_1 + a_2 + \ldots + a_n
\]

If the sequence \( \{S_n\}_{n=1}^{\infty} \) is convergent, with limit \( S \), then we say the infinite series \( \sum_{n=1}^{\infty} a_n \) is convergent and write \( \sum_{n=1}^{\infty} a_n = S \).
s is called the sum of the series. If \( \{s_n\} \) is divergent, we say \( \sum_{n=1}^{\infty} a_n \) is divergent.

Very important examples

Let \( a \to 0 \) and \( r > 0 \). Consider the sequence

\[
\{ar^n\}_{n=1}^{\infty} = \{a, ar, ar^2, ar^3, \ldots\}
\]

Consider \( s_n = a + ar + \ldots + ar^{n-1} \). Observe that if \( r=1 \) then \( s_n = na \). In this case \( \{s_n\}_{n=1}^{\infty} \) is divergent so \( \sum_{n=1}^{\infty} ar^{n-1} \) is divergent. Assume \( r \neq 1 \). Note that

\[
s_n = ar + ar^2 + \ldots + ar^n \implies
s_n = \frac{a(r^{n+1} - 1)}{r-1}
\]

If \( r > 1 \) \( \implies \{r^n\}_{n=1}^{\infty} \) divergent \( \implies \{s_n\}_{n=1}^{\infty} \) divergent \( \implies \sum_{n=1}^{\infty} ar^{n-1} \) divergent.

If \( 0 < r < 1 \) \( \implies \{r^n\}_{n=1}^{\infty} \) converged to 0 \( \implies \)

\( \{s_n\}_{n=1}^{\infty} \) converged to \( \frac{-a}{r-1} = \frac{a}{1-r} \).
\[
\sum_{n=1}^{\infty} ar^n \text{ convergent with sum } \frac{a}{1-r}.
\]

In example (1) we have \( a = \frac{1}{2} \) and \( r = \frac{1}{2} \) so

\[
\frac{a}{1-r} = \frac{\left( \frac{1}{2} \right)}{\left( \frac{1}{2} \right)} = 1 \quad \text{as we observed!}
\]

This class of infinite series are called geometric series.

So the main trick to understanding \( \sum_{n=1}^{\infty} a_n \) is understanding the sequence of partial sums \( \{s_n\}_{n=1}^{\infty} \).

Interesting example: The harmonic series is the infinite series

\[
\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots
\]

Is this series convergent? It's tempting to guess yes, because \( \frac{1}{n} \to 0 \) as \( n \to \infty \). However, this is not the case as the following observations show:

\[
\begin{align*}
S_2 &= 1 + \frac{1}{2} \\
S_4 &= 1 + \frac{1}{2} + \left( \frac{1}{3} + \frac{1}{4} \right) > 1 + \frac{1}{2} + \left( \frac{1}{4} + \frac{1}{4} \right) = 2 \\
S_8 &= 1 + \frac{1}{2} + \left( \frac{1}{3} + \frac{1}{4} \right) + \left( \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} \right) > 1 + \frac{1}{2} + \left( \frac{1}{4} + \frac{1}{4} \right) + \left( \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} \right) = 1 + \frac{3}{2}
\end{align*}
\]
Carrying on in this fashion we get

\[ S_{16} > 1 + \frac{4}{2}, \quad S_{32} > 1 + \frac{5}{2}, \quad S_{64} > 1 + \frac{6}{2}. \]

and in general \( S_{2^n} > 1 + \frac{n}{2}. \)

This shows that \( S_{2^n} \to \infty \) as \( n \to \infty \), hence \( \{S_n\} \) is divergent! Hence \( 1 + \frac{1}{2} + \frac{1}{3} + \ldots \)

diverges!

**Fact**: If \( \sum_{n=1}^{\infty} a_n \) converges, then \( \lim_{n \to \infty} a_n = 0. \)

However, if we just know that \( \lim_{n \to \infty} a_n = 0 \), this does not imply that \( \sum_{n=1}^{\infty} a_n \) converges.

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**Conclusion**: Spotting whether \( \sum_{n=1}^{\infty} a_n \) is convergent by looking at \( \{a_n\} \) is subtle and not easy. Most of this chapter is about understanding when series converge and diverge.

Go and read Theorem 8 on page 709 for some nice properties of convergent series under Rolle's and
scaling. Then carefully go through example 8.