Math 1B : Calculus (Fall 2014)

Lecture 16 : <u>Series</u>

Let $\{a_n\}_{n=1}^{\infty}$ be a sequence. Does it make sense to talk about the sum

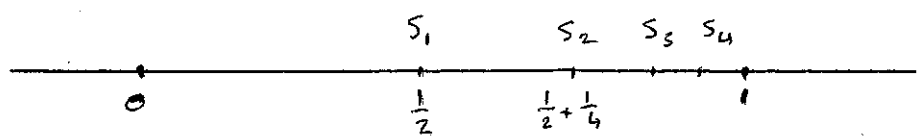
$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + a_4 + \dots \quad ?$$

<u>Examples</u> 1) Let $\{a_n\}_{n=1}^{\infty}$ be the sequence with $a_n = \frac{1}{2^n}$.

i.e. $\left\{\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \dots\right\}$

Let $s_n = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots + \frac{1}{2^n}$

What happens as $n \to \infty$?



From the picture we can guess the following:

$$\lim_{n\to\infty} \{s_n\} = 1$$

So even though it doesn't make sense to literally add together infinitely many terms, in this case it is sensible to write $\sum_{n=1}^{\infty} a_n = 1$ //

2) Let $\{a_n\}_{n=1}^{\infty}$ be the sequence $a_n = n$, i.e. $\{1, 2, 3, 4, \ldots\}$. If we do the same thing here then

$$S_n = 1 + 2 + 3 + \ldots + n.$$

Clearly $S_n \geq n$ for all $n \geq 1$, hence $\lim_{n \to \infty} \{S_n\} = \infty$. So $\sum_{n=1}^{\infty} a_n$ doesn't have any meaning as a number in this case.

These observations inspire the following definition.

<u>Definition</u>

Let $\{a_n\}_{n=1}^{\infty}$ be a sequence. We define the <u>n^{th} partial sum</u> to be

$$S_n = \sum_{i=1}^{n} a_i = a_1 + a_2 + \ldots + a_n$$

If the sequence $\{S_n\}_{n=1}^{\infty}$ is convergent, with limit S, then we say the <u>infinite series</u> $\sum_{n=1}^{\infty} a_n$ is convergent and write $\sum_{n=1}^{\infty} a_n = S$.

S is called the <u>sum</u> of the series. If $\{S_n\}$ is <u>divergent</u>, we say $\sum_{n=1}^{\infty} a_n$ is <u>divergent</u>.

<u>Very important examples</u>

Let $a \neq 0$ and $r > 0$. Consider the sequence

$$\{ar^{n-1}\}_{n=1}^{\infty} = \{a, ar, ar^2, ar^3, \dots\}$$

Consider $S_n = a + ar + \dots + ar^{n-1}$. Observe that if $r=1$ then $S_n = na$. In this case $\{S_n\}_{n=1}^{\infty}$ is divergent so $\sum_{n=1}^{\infty} ar^{n-1}$ is divergent. Assume $r \neq 1$. Note that

$$rS_n = ar + ar^2 + \dots + ar^n \Rightarrow$$

$$rS_n - S_n = ar^n - a \Rightarrow S_n = \frac{a(r^n - 1)}{r-1}$$

If $r > 1 \Rightarrow \{r^n\}_{n=1}^{\infty}$ divergent $\Rightarrow \{S_n\}_{n=1}^{\infty}$ divergent $\Rightarrow \sum_{n=1}^{\infty} ar^{n-1}$ divergent.

If $0 \leq r < 1 \Rightarrow \{r^n\}_{n=1}^{\infty}$ convergent to $0 \Rightarrow$ $\{S_n\}_{n=1}^{\infty}$ convergent to $\frac{-a}{r-1} = \frac{a}{1-r} \Rightarrow$

$$\sum_{n=1}^{\infty} ar^n \text{ convergent with sum } \frac{a}{1-r}.$$

In example (1) we have $a = \frac{1}{2}$ and $r = \frac{1}{2}$ so

$$\frac{a}{1-r} = \frac{\left(\frac{1}{2}\right)}{\left(\frac{1}{2}\right)} = 1 \quad \text{as we observed!}$$

This class of infinite series are called <u>geometric series</u>

So the main trick to understanding $\sum_{n=1}^{\infty} a_n$ is understanding the sequence of partial sums $\{s_n\}_{n=1}^{\infty}$.

Interesting example: The harmonic series is the infinite series

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots$$

Is this series convergent? It's tempting to guess yes, because $\frac{1}{n} \to 0$ as $n \to \infty$. However this is not the case as the following observations show:

$$s_2 = 1 + \frac{1}{2}$$

$$s_4 = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) > 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) = 2$$

$$s_8 = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) >$$

$$1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) = 1 + \frac{3}{2}$$

Carrying on in this fashion we get

$$S_{16} > 1 + \frac{4}{2}, \quad S_{32} > 1 + \frac{5}{2}, \quad S_{64} > 1 + \frac{6}{2}.$$

and in general $S_{2^n} > 1 + \frac{n}{2}$.

This shows that $S_{2^n} \to \infty$ as $n \to \infty$, hence $\{S_n\}$ is <u>divergent!</u> Hence $1 + \frac{1}{2} + \frac{1}{3} + \dots$ diverges!

<u>Fact</u> If $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} \{a_n\} = 0$.

However, if we just know that $\lim_{n\to\infty} \{a_n\} = 0$, this <u>does not</u> imply that $\sum_{n=1}^{\infty} a_n$ converges.

<u>Conclusion</u>: Spotting whether $\sum_{n=1}^{\infty} a_n$ is convergent by looking at $\{a_n\}$ is subtle and not easy. Most of this chapter is about understanding when series converge and diverge.

Go and read Theorem 8 on page 709 for some nice properties of convergent series under addition and

scaling. Then carefully go through example 8.